CHAPTER 4

Matrices

4A Matrix Operations

4-1 Matrices and Data

4-2 Multiplying Matrices

4-3 Using Matrices to Transform Geometric Figures

CONCEPT CONNECTION

4B Using Matrices to Solve Systems

4-4 Determinants and Cramer's Rule

4-5 Matrix Inverses and Solving Systems

Lab Use Spreadsheets with Matrices

4-6 Row Operations and Augmented Matrices

Ext Networks and Matrices

CONCEPT CONNECTION

Microprocessors, like those inside computers, use matrices to store, analyze, and calculate large quantities of data.

Silicon Valley, CA

ARE YOU READY?

Vocabulary

Match each term on the left with a definition on the right.

1. radius
2. dependent system
3. inconsistent system
4. transformation

A. an operation that can be performed in either order, as in $a + b = b + a$ and $ab = ba$

B. the distance from the center of a circle to the circle

C. A system of equations or inequalities that has no solution

D. A change in the position, size, or shape of a figure or graph

E. A system of equations that has infinitely many solutions

Add and Subtract Integers

Simplify each expression.

5. $2 + 7 + (-10)$ **6.** $-8 + 14 + (-3)$ **7.** $-2 + (-3) + (-5)$

8. $-9 + 15 - 7 + 1$ **9.** $20 - (-5) + (-3) - 2$ **10.** $9 + 8 - 7 + 5 - (-3) + 2$

Multiply and Divide Integers

Multiply or divide.

11. $-18 \div 9$ **12.** $-6(-1)$ **13.** $16(-2)$ **14.** $-15 \div (-3)$

Order of Operations

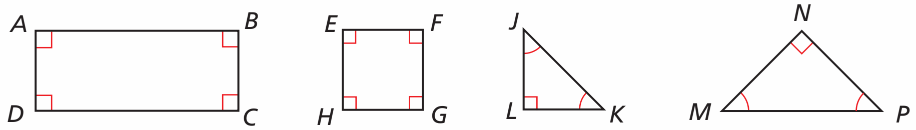
Simplify each expression.

15. $2(0.5) + 2(0.6)$ **16.** $0(6.7) + 1(0.3) - 5(2) - 3(8)$

17. $3(2 + 7 + 0) - 5(3 + 6 + 4)$ **18.** $4(3 - 6 + 2) - 5(2 + 0 - 1)$

Identify Similar Figures

19. Identify which figures are similar.



Find Missing Measures in Similar Figures

20. $\triangle ABC$ is similar to $\triangle DEF$. $m\angle FDE = 35°$. What other angle has a measure of $35°$?

21. $\triangle FGH$ is similar to $\triangle JKL$. $JL = 12$, $GH = 12$, and $FH = 8$. Find KL.

CHAPTER 4

Unpacking the Standards

The information below "unpacks" the standards. The Academic Vocabulary is highlighted and defined to help you understand the language of the standards. Refer to the lessons listed after each standard for help with the math terms and phrases. The Chapter Concept shows how the standard is applied in this chapter.

California Standard	Academic Vocabulary	Chapter Concept
2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or **with matrices.** (Lessons 4-4, 4-5, 4-6)	**system** a combination of parts that forms a whole **matrices** (singular: matrix) a rectangular array of numbers ***Example:*** $\begin{bmatrix} 12 & 8 & 7 \\ 6 & 23 & 14 \end{bmatrix}$	You solve systems of equations using matrices.
Review of 7MG3.2 Understand and use coordinate graphs to plot simple figures, determine lengths and areas related to them, **and determine their image under translations and reflections.** (Connecting)	**plot** draw on a graph **image** a shape that results from a transformation of a figure known as the preimage	You transform figures in the coordinate plane.

Reading and Writing Math

CHAPTER 4

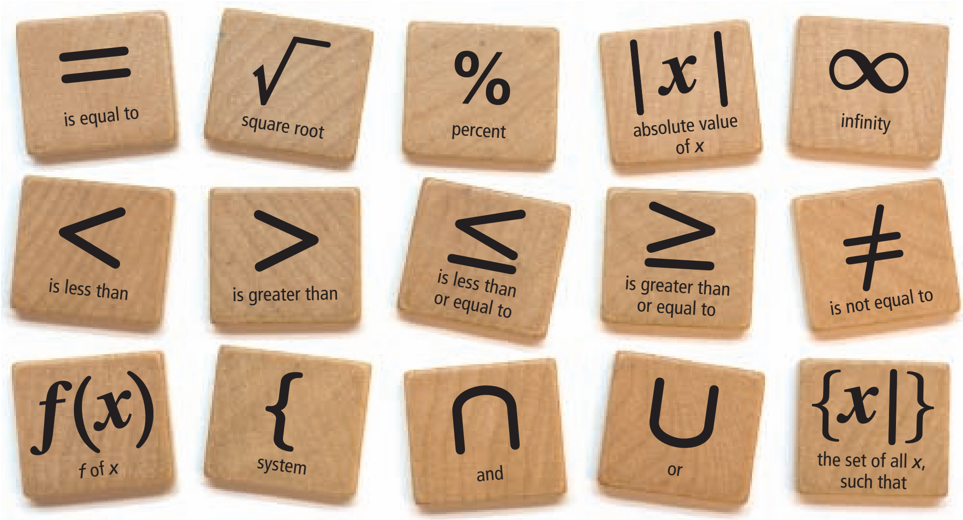
Reading Strategy: Read and Interpret Math Symbols

Interpreting math symbols is a necessary skill that you need in order to comprehend new material. As you study each lesson in this textbook, read aloud the expressions involving symbols and notations. This practice will help you become proficient at translating symbols into words.

Common Math Symbols

Inequality Symbols

Function and Set Notation



In Algebra, symbols are used to communicate information. As you study each lesson, read aloud expressions involving symbols and expressions. This can help you translate symbols into words.

Expressions	Words
$f(x) = \sqrt{16x} - 4$	*f* of *x* is equal to the square root of 16 times *x*, minus 4.
$\frac{\lvert x - 15 \rvert}{6} \leq 12$	The absolute value of the quantity *x* minus 15, divided by 6, is less than or equal to 12.
$\{x \mid x \leq -19 \cup x > 8\}$	The set of all numbers *x* such that *x* is less than or equal to negative 19 OR *x* is greater than 8
$\begin{cases} y \leq -4x + 8 \\ y > x - 6 \end{cases}$	The system of inequalities containing "*y* is less than or equal to negative 4*x* plus 8" and "*y* is greater than *x* minus 6"

Try This

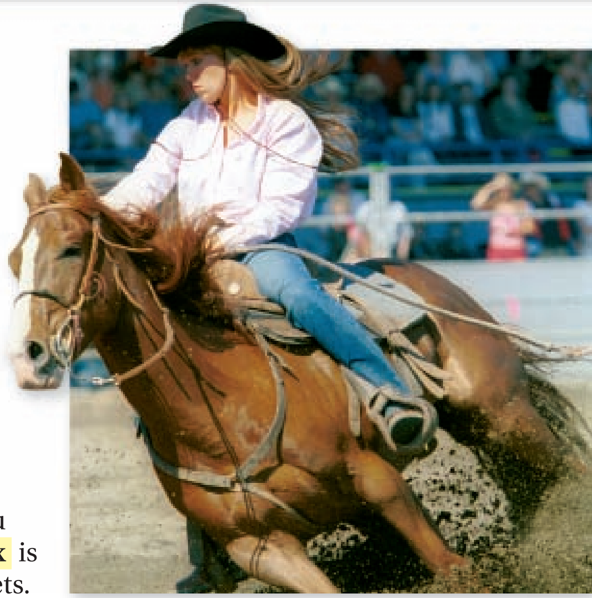
Translate these mathematical expressions into words.

1. $\{x \mid x \geq -7 \cup x \leq -1\}$
2. $f(y) = \lvert 15y \rvert + \frac{y}{2}$
3. $\begin{cases} y = 2x + 3 \\ y = x \end{cases}$
4. $[-5, \infty)$

Rewrite the statement as an algebraic expression.

5. The set of all numbers *x* such that *x* is between negative 8 and 10.

4-1 Matrices and Data



Objectives

Use matrices to display mathematical and real-world data.

Find sums, differences, and scalar products of matrices.

Vocabulary

matrix
dimensions
entry
address
scalar

California Standards

Preparation for 2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or **with matrices.**

Who uses this?

Rodeo scorekeepers may use matrices to determine scores of participants for events such as barrel racing.

The table shows the top scores for girls in barrel racing at the 2004 National High School Rodeo finals. The data can be presented in a table or a spreadsheet as rows and columns of numbers. You can also use a *matrix* to show table data. A **matrix** is a rectangular array of numbers enclosed in brackets.

2004 National High School Rodeo Finals—Barrel Racing Scores

Participant	First Ride	Second Ride	Third Ride
Sierra Thomas (UT)	16.781	16.29	17.318
Kelly Allen (TX)	16.206	16.606	17.668

Matrix A has two rows and three columns. A matrix with m rows and n columns has **dimensions** $m \times n$, read "m by n," and is called an $m \times n$ matrix. A has dimensions 2×3. Each value in a matrix is called an **entry** of the matrix.

$$A = \begin{bmatrix} 16.781 & 16.29 & 17.318 \\ 16.206 & 16.606 & 17.668 \end{bmatrix} \begin{matrix} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \end{matrix}$$

Column 1 Column 2 Column 3

The **address** of an entry is its location in a matrix, expressed by using the lowercase matrix letter with the row and column number as subscripts. The score 16.206 is located in row 2 column 1, so a_{21} is 16.206.

$$A = \begin{bmatrix} 16.781 & 16.29 & 17.318 \\ 16.206 & 16.606 & 17.668 \end{bmatrix}$$

a_{21} → 16.206

EXAMPLE 1 Displaying Data in Matrix Form

Use the packaging data for the costs of the packages given.

Cost of 4-Inch Cubic Box ($)

	Plastic	Paper
Total Cost	0.48	0.72
Cost per in^2	0.005	0.0075
Cost per in^3	0.0075	0.01125

a. Display the data in matrix form.

$$C = \begin{bmatrix} 0.48 & 0.72 \\ 0.005 & 0.0075 \\ 0.0075 & 0.01125 \end{bmatrix}$$

b. What are the dimensions of C?

C has three rows and two columns, so it is a 3×2 matrix.

c. What is the entry at c_{12}? What does it represent?

The entry at c_{12}, in row 1 column 2, is 0.72. It is the total cost of a 4 in. paper box.

d. What is the address of the entry 0.005?

The entry 0.005 is at c_{21}.

CHECK IT OUT! **Use matrix M to answer the questions below.**

$$M = \begin{bmatrix} 2 & 1 & 5 & 0 \\ 1 & 5 & 0 & 9 \\ 2 & 11 & 4 & 12 \end{bmatrix}$$

1a. What are the dimensions of M?

1b. What is the entry at m_{32}?

1c. The entry 0 appears at what two addresses?

Corresponding entries in two or more matrices are entries with the same address, such as a_{32} and b_{32} in matrices A and B.

Know it! Note

Adding and Subtracting Matrices

WORDS	NUMBERS	ALGEBRA
To add or subtract two matrices, add or subtract the corresponding entries.	$\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 10 \end{bmatrix} = \begin{bmatrix} 6 & 12 \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \end{bmatrix}$

You can add or subtract two matrices only if they have the same dimensions.

✔ Same Dimensions

$$\begin{bmatrix} 1 & 2 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$$

✘ Different Dimensions

$$\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix}$$

EXAMPLE 2 Finding Matrix Sums and Differences

$$A = \begin{bmatrix} 4 & -2 \\ -3 & 10 \\ 2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 & -5 \\ 3 & 2 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 \\ 0 & -9 \\ -5 & 14 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 & -3 \\ 3 & 0 & 10 \end{bmatrix}$$

Add or subtract, if possible.

A $A + C$

Add each corresponding entry.

$$A + C = \begin{bmatrix} 4 & -2 \\ -3 & 10 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 0 & -9 \\ -5 & 14 \end{bmatrix} = \begin{bmatrix} 4 + 3 & -2 + 2 \\ -3 + 0 & 10 + (-9) \\ 2 + (-5) & 6 + 14 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ -3 & 1 \\ -3 & 20 \end{bmatrix}$$

B $C - A$

Subtract each corresponding entry.

$$C - A = \begin{bmatrix} 3 & 2 \\ 0 & -9 \\ -5 & 14 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -3 & 10 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3 - 4 & 2 - (-2) \\ 0 - (-3) & -9 - 10 \\ -5 - 2 & 14 - 6 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 3 & -19 \\ -7 & 8 \end{bmatrix}$$

C $C + B$

C is a 3×2 matrix, and B is a 2×3 matrix. Because C and B do not have the same dimensions, they cannot be added.

CHECK IT OUT! **Add or subtract, if possible.**

2a. $B + D$ **2b.** $B - A$ **2c.** $D - B$

You know that multiplication is repeated addition. The same is true for matrices.

For example, let $E = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$.

$$E + E = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2+2 & 0+0 \\ 1+1 & 5+5 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(0) \\ 2(1) & 2(5) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 10 \end{bmatrix}$$

$E + E$ can be written as $2E$. You can multiply a matrix by a number, called a **scalar**. To find the product of a scalar and a matrix, or the *scalar product,* multiply each entry by the scalar.

$$2\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(0) \\ 2(1) & 2(5) \end{bmatrix}$$

EXAMPLE 3 *Business Application*

A ticket service marks up prices on tickets to rodeos and other events by 150%. Use a scalar product to find the marked-up prices.

Rodeo Ticket Prices		
Days	**Plaza**	**Balcony**
1–2	$60	$35
3–8	$50	$28
9–10	$80	$45

You can multiply by 1.5 and add to the original numbers.

$$\begin{bmatrix} 60 & 35 \\ 50 & 28 \\ 80 & 45 \end{bmatrix} + 1.5\begin{bmatrix} 60 & 35 \\ 50 & 28 \\ 80 & 45 \end{bmatrix}$$

$$= \begin{bmatrix} 60 & 35 \\ 50 & 28 \\ 80 & 45 \end{bmatrix} + \begin{bmatrix} 90 & 52.5 \\ 75 & 42 \\ 120 & 67.5 \end{bmatrix} = \begin{bmatrix} 150 & 87.5 \\ 125 & 70 \\ 200 & 112.5 \end{bmatrix}$$

The marked-up prices are shown below.

Ticket Service Prices		
Days	**Plaza**	**Balcony**
1–2	$150	$87.50
3–8	$125	$70.00
9–10	$200	$112.50

Helpful Hint

In Example 3, a markup of 150% is the same as an increase of 150%.

CHECK IT OUT! **3.** Use a scalar product to find the prices if a 20% discount is applied to the ticket service prices.

EXAMPLE 4 Simplifying Matrix Expressions

$$A = \begin{bmatrix} 4 & -2 \\ -3 & 10 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 & -5 \\ 3 & 2 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 \\ 0 & -9 \end{bmatrix} \quad D = \begin{bmatrix} -6 & 3 & 8 \end{bmatrix}$$

A **Evaluate $2A - 3B$, if possible.**

$$2\begin{bmatrix} 4 & -2 \\ -3 & 10 \end{bmatrix} - 3\begin{bmatrix} 4 & -1 & -5 \\ 3 & 2 & 8 \end{bmatrix}$$

A and B do not have the same dimensions; they cannot be subtracted after the scalar products are found.

Helpful Hint

A matrix bracket is a grouping symbol. So in an expression like $C - 2A$, you distribute -2 to all of the entries in A before adding, just as you do with numbers.

B **Evaluate $C - 2A$, if possible.**

$$= \begin{bmatrix} 3 & 2 \\ 0 & -9 \end{bmatrix} - 2\begin{bmatrix} 4 & -2 \\ -3 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 0 & -9 \end{bmatrix} + \begin{bmatrix} -2(4) & -2(-2) \\ -2(-3) & -2(10) \end{bmatrix}$$ *Multiply each entry by −2.*

$$= \begin{bmatrix} 3 & 2 \\ 0 & -9 \end{bmatrix} + \begin{bmatrix} -8 & 4 \\ 6 & -20 \end{bmatrix} = \begin{bmatrix} -5 & 6 \\ 6 & -29 \end{bmatrix}$$

CHECK IT OUT! **Evaluate, if possible.**

4a. $3B + 2C$ **4b.** $2A - 3C$ **4c.** $D + 0.5D$

Some properties of equality also apply to matrices.

Know it! Note

Properties of Equality for Matrices

WORDS	NUMBERS	ALGEBRA
Commutative Property Matrix addition is commutative.	$\begin{bmatrix} 7 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 2 \\ 3 & 4 \end{bmatrix}$	$A + B = B + A$
Associative Property Matrix addition is associative.	$\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)$	$A + B + C = (A + B) + C = A + (B + C)$
Additive Identity The *zero matrix* is the *additive identity matrix* O.	$\begin{bmatrix} 7 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 3 & 4 \end{bmatrix}$	$A + O = A$
Additive Inverse The *additive inverse* of matrix A contains the opposite of each entry in matrix A.	$\begin{bmatrix} 5 & -2 \\ -6 & 9 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 6 & -9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	If $A + B = O$, then A and B are additive inverses.

THINK AND DISCUSS

1. Find the possible dimensions of a matrix that contains eight entries.

2. Describe a matrix operation that reverses the signs of every entry.

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer. Give examples for matrices and real numbers.

Property or Operation	Real Numbers	Matrices
Addition		
Subtraction		
Multiplication by a number		

4-1 Exercises

California Standards
Preparation for 2.0;
Review of 7AF1.1

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Homework Help Online
KEYWORD: MB7 4-1
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

1. **Vocabulary** The value at a particular place in a matrix is an __?__ . (*address* or *entry*)

SEE EXAMPLE 1 p. 246

2. Kade, Bo, and Tanner record their ticket-selling activities for a fund-raising carnival.

Carnival Ticket Prices			
Student	**Single Tickets**	**Ticket Packages**	**Total Collected**
Kade	39	15	$114
Bo	103	8	$143
Tanner	13	25	$138

a. Display the data in the form of a matrix T.
b. What are the dimensions of T?
c. What is the entry at t_{13}? What does it represent?
d. What is the address of the entry 143?

SEE EXAMPLE 2 p. 247

Use the following matrices for Exercises 3–6. Add or subtract, if possible.

$$A = \begin{bmatrix} 1.5 & 3.8 & 3 \\ -1.2 & 2.4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 4 & 1 \\ 0 & -2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1.1 & 6 \\ 4 & 0 & 1 \\ 1 & 2.3 & 1 \end{bmatrix}$$

3. $A + B$
4. $B - C$
5. $B - A$
6. $B + A$

SEE EXAMPLE 3 p. 248

7. **Consumer** The table shows prices for three types of clothing. Use a scalar product to find the price with 8.25% sales tax on each item.

Cost of Athletic Clothing ($)			
	Plain	**Team Logo**	**Individualized**
T-shirt	9.00	13.00	14.00
Shorts	6.00	9.50	11.00
Jogging Pants	15.00	21.00	23.00

SEE EXAMPLE 4 p. 248

Use the following matrices for Exercises 8–11. Evaluate, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -1 & 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 4 & 1 \\ 0 & -2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 & 6 \\ 4 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

8. $3B$
9. $\frac{1}{2}C$
10. $A - 2B$
11. $2C - A$

PRACTICE AND PROBLEM SOLVING

12. Use the data to answer the questions.
a. Display the data as a matrix, P.
b. What are the dimensions of P?
c. What is the entry at p_{32}? What does it represent?
d. What is the address of the entry 385.98?

Travel Options			
	Airfare	**Hotel**	**Car Rental**
Deluxe	425.50	398.00	65.99
Business	385.98	245.50	45.90
Economy	275.12	103.25	29.50

Independent Practice	
For Exercises	See Example
12	1
13–16	2
17	3
18–21	4

Extra Practice

Skills Practice p. S10
Application Practice p. S35

Use the following matrices for Exercises 13–16. Add or subtract, if possible.

$$D = \begin{bmatrix} 5.1 & 2.5 \\ -2 & 0 \\ 0 & 1.5 \end{bmatrix} \quad E = \begin{bmatrix} 3.2 & -1 \\ -1.5 & 2.4 \end{bmatrix} \quad F = \begin{bmatrix} -4.2 & -1 \\ 2.2 & 0 \end{bmatrix}$$

13. $F - E$ **14.** $D + E$ **15.** $D + F$ **16.** $E + F$

17. College The following table shows estimated college costs in 2004.

Estimated College Costs (per Year) in 2004			
	Private School	In-State Public School	Out-of-State Public School
Cost ($)	27,677	12,841	19,188

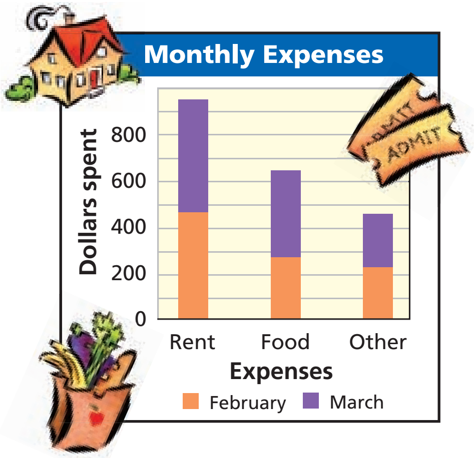
Costs are expected to increase 5% per year. Use a scalar product to find the estimated costs for each type of college in 2005.

Use the following matrices for Exercises 18–21. Evaluate, if possible.

$$G = \begin{bmatrix} 5 & 2 \\ -2 & 0 \\ 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 0 & -1 \\ -1 & 2 \\ 0 & 2 \end{bmatrix} \quad J = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} \quad K = \begin{bmatrix} 2 & 3 \\ 3 & -1 \\ 5 & 0 \end{bmatrix}$$

18. $2G$ **19.** $\frac{1}{2}(H + J)$ **20.** $2K - G$ **21.** $J - 0.3G$

22. Estimation Trey recorded his total expenses for February and March in a spreadsheet and graphed the results. Write 3×1 matrices to represent his expenses in February and March, and show the matrix sum for his total expenses.



23. Geometry The matrix $R = \begin{bmatrix} 2 & 2.5 \\ 3 & 3.5 \end{bmatrix}$ shows the radii of four circles.

a. Write the matrix operation that gives the related circumferences.

b. Is there an addition or scalar-multiplication matrix operation that could show the related areas of the circles? Explain.

Critical Thinking **Tell whether each statement is sometimes, always, or never true.**

24. If matrices A and B have an equal number of entries, then $A + B$ is defined.

25. If matrices A and B have a different number of entries, then $A + B$ is defined.

26. If matrices A and B each have four rows and three columns, then $A + B$ is defined.

27. If $A + B$ is defined, then $A - B$ is defined.

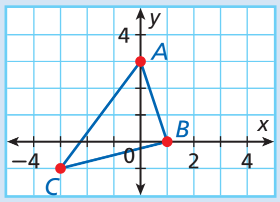
Concept Connection

28. This problem will prepare you for the Concept Connection on page 268.

a. Place the vertices of the triangle in a matrix so that the x-coordinates are in row 1 and the y-coordinates are in row 2.

b. Use a matrix operation to add 3 to each x-coordinate and 1 to each y-coordinate.

c. Draw a new triangle using the new coordinates. Describe the new triangle.



29. Solve for a, b, and c in the matrix equation. $\begin{bmatrix} 3 & a \\ -2 & -8 \end{bmatrix} + \begin{bmatrix} 11 & -4 \\ b & 12 \end{bmatrix} = \begin{bmatrix} 14 & -10 \\ 9 & c \end{bmatrix}$

30. **///ERROR ANALYSIS///** Explain the error. $\begin{bmatrix} 2 & 8 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 3 & 0 \\ 4 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 0 \\ 8 & 8 & 9 \end{bmatrix}$

31. **Write About It** Is subtraction of matrices commutative? Give an example to support your answer.

Standardized Test Prep

32. $P = \begin{bmatrix} 1 & 0.1 & 2 \\ 1.5 & 2.1 & 0 \end{bmatrix}$ $Q = \begin{bmatrix} 2 & 0.4 & 6 \\ 6 & 6.4 & 0 \end{bmatrix}$. Which expression results in $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$?

Ⓐ $2Q - \frac{1}{2}P$ Ⓑ $Q - 2P$ Ⓒ $P - 2Q$ Ⓓ $2P - \frac{1}{2}Q$

33. For an $m \times n$ matrix E, which statement is always true?

Ⓕ It has $m \cdot n$ entries. Ⓗ It has $m + n$ entries.

Ⓖ It has an entry e_{nm}. Ⓙ It has m columns and n rows.

34. Solve for w: $8\begin{bmatrix} 12 & 8 \\ 2 & 7 \end{bmatrix} = w\begin{bmatrix} 48 & 32 \\ 8 & 28 \end{bmatrix}$.

Ⓐ 0.25 Ⓑ 0.5 Ⓒ 2 Ⓓ 4

35. **Gridded Response** Solve for x: $\begin{bmatrix} 2 & -2 \end{bmatrix} - 2\begin{bmatrix} 5 & -x \end{bmatrix} = \begin{bmatrix} -8 & -1 \end{bmatrix}$.

CHALLENGE AND EXTEND

36. **Critical Thinking** If the number of entries in a matrix is a prime number, what must be true about the dimensions of the matrix? Explain.

37. Explain why, for any two $m \times n$ matrices A and B, $A - B$ is equivlent to $A + (-B)$.

38. In *magic squares* like those shown, the rows, columns, and diagonals all have the same sum. Is the sum of the two magic squares also a magic square? Explain.

8	3	4
1	5	9
6	7	2

4	18	8
14	10	6
12	2	16

39. $3\begin{bmatrix} 2 & -1 \\ 0 & -4 \end{bmatrix} - 2B = \begin{bmatrix} 1 & 5 \\ -2 & 2 \end{bmatrix}$. Find B.

SPIRAL REVIEW

Write an algebraic expression to represent each situation. *(Lesson 1-4)*

40. the perimeter of a triangle with side lengths that are consecutive even integers

41. the total number of raffle tickets sold if 20 people each sold n tickets

42. **Money** Nyla has 36 nickels and dimes. She has twice as many dimes as nickels. How much money does Nyla have? *(Lesson 2-1)*

Determine if the given point is a solution of the system of equations. *(Lesson 3-1)*

43. $(2, -2)$ $\begin{cases} x - y = 4 \\ 5x + 6y = 2 \end{cases}$

44. $(4.5, 2)$ $\begin{cases} y = 2 \\ 2x - 4y = 1 \end{cases}$

4-2 Multiplying Matrices

Objectives
Understand the properties of matrices with respect to multiplication.

Multiply two matrices.

Vocabulary
matrix product
square matrix
main diagonal
multiplicative identity matrix

Who uses this?
Skateboard shop owners can use matrices to find the value of their inventory. (See Example 3.)

In Lesson 4-1, you multiplied matrices by a number called a *scalar*. You can also multiply matrices together. The product of two or more matrices is the **matrix product**. The following rules apply when multiplying matrices.

- Matrices A and B can be multiplied only if the number of columns in A equals the number of rows in B.
- The product of an $m \times n$ and an $n \times p$ matrix is an $m \times p$ matrix.

Helpful Hint
The CAR key:
Columns (of *A*)
As
Rows (of *B*)
or matrix product *AB* won't even start

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 4 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 & 3 & 8 \\ 9 & 5 & 2 & 0 \\ 0 & 1 & 6 & 7 \end{bmatrix}$$

A	B		AB
2×3	3×4	=	2×4 matrix

columns = rows

$$C = \begin{bmatrix} 3 & 5 \\ 4 & 1 \\ 5 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 3 & 3 & 8 & 4 \\ 9 & 5 & 2 & 0 & 6 \\ 0 & 1 & 6 & 7 & 2 \end{bmatrix}$$

C	D	
3×2	3×5	✘ CD is not defined $(2 \neq 3)$

columns ≠ rows

An $m \times n$ matrix A can be identified by using the notation $A_{m \times n}$.

EXAMPLE 1

Identifying Matrix Products

California Standards
Preparation for 🔑 2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or **with matrices.**

Tell whether each product is defined. If so, give its dimensions.

A $P_{2 \times 5}$ and $Q_{5 \times 3}$; PQ

P	Q	PQ
2×5	5×3	$= 2 \times 3$ matrix

The inner dimensions are equal $(5 = 5)$, so the matrix product is defined. The dimensions of the product are the outer numbers, 2×3.

B $R_{4 \times 3}$ and $S_{4 \times 5}$; RS

R	S
4×3	4×5

The inner dimensions are not equal $(3 \neq 4)$, so the matrix product is not defined. ✘

CHECK IT OUT! **Use the matrices in Example 1. Tell whether each product is defined. If so, give its dimensions.**

1a. *QP* **1b.** *SR* **1c.** *SQ*

Just as you look across the columns of A and down the rows of B to see if a product AB exists, you do the same to find the entries in a matrix product.

Know it! Note

Multiplying Matrices

WORDS	NUMBERS	ALGEBRA
In a matrix product $P = AB$, each element p_{ij} is the sum of the products of consecutive entries in row i in matrix A and column j in matrix B.	$P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix}$	$P = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} = \begin{bmatrix} a_1c_1 + a_2d_1 & a_1c_2 + a_2d_2 \\ b_1c_1 + b_2d_1 & b_1c_2 + b_2d_2 \end{bmatrix}$

EXAMPLE 2

Finding the Matrix Product

Find each product, if possible. $A = \begin{bmatrix} 0 & 4 & 9 \\ -3 & 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 1 \\ -2 & 7 \\ 6 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 11 & -1 \\ 12 & 10 \end{bmatrix}$

A AB

Check the dimensions. A is 2×3, B is 3×2. AB is defined and is 2×2.

Multiply row 1 of A and column 1 of B as shown. Place the result in ab_{11}.

$$AB = \begin{bmatrix} 0 & 4 & 9 \\ -3 & 3 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ -2 & 7 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 46 & ? \\ ? & ? \end{bmatrix} \leftarrow 0(5) + 4(-2) + 9(6)$$

Multiply row 1 of A and column 2 of B. Place the result in ab_{12}.

$$\begin{bmatrix} 0 & 4 & 9 \\ -3 & 3 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ -2 & 7 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 46 & 28 \\ ? & ? \end{bmatrix} \leftarrow 0(1) + 4(7) + 9(0)$$

Multiply row 2 of A and column 1 of B. Place the result in ab_{21}.

$$\begin{bmatrix} 0 & 4 & 9 \\ -3 & 3 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ -2 & 7 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 46 & 28 \\ -9 & ? \end{bmatrix} \leftarrow -3(5) + 3(-2) + 2(6)$$

Multiply row 2 of A and column 2 of B. Place the result in ab_{22}.

$$\begin{bmatrix} 0 & 4 & 9 \\ -3 & 3 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ -2 & 7 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 46 & 28 \\ -9 & 18 \end{bmatrix} \quad AB = \begin{bmatrix} 46 & 28 \\ -9 & 18 \end{bmatrix}$$

$$-3(1) + 3(7) + 2(0)$$

Caution!

Notice that AB and BA are different products.

The Commutative Property does not hold for multiplication of matrices!

B BA

Check the dimensions. B is 3×2, and A is 2×3, so the product is defined and is 3×3.

$$BA = \begin{bmatrix} 5(0) + 1(-3) & 5(4) + 1(3) & 5(9) + 1(2) \\ -2(0) + 7(-3) & -2(4) + 7(3) & -2(9) + 7(2) \\ 6(0) + 0(-3) & 6(4) + 0(3) & 6(9) + 0(2) \end{bmatrix} = \begin{bmatrix} -3 & 23 & 47 \\ -21 & 13 & -4 \\ 0 & 24 & 54 \end{bmatrix}$$

C AC

Check the dimensions: $2 \times 3 \quad 2 \times 2$. The product is not defined. The matrices cannot be multiplied in this order.

CHECK IT OUT! **Find the product, if possible.**

2a. BC **2b.** CA

Businesses can use matrix multiplication to find total revenues, costs, and profits.

EXAMPLE 3 *Inventory Application*

A skateboard kit comes in two styles. Two stores have inventories as shown in the first table. Find the total cost of the skateboards for each store.

Skateboard Kit Inventory

	Complete	Super Complete
Store 1	14	10
Store 2	7	8

Skateboard Kit Profits

	Revenue ($)	Store Cost ($)	Profit ($)
Complete	89	44	45
Super Complete	119	58	61

Use a product matrix to find the revenue, cost, and profit for each store.

$$\begin{bmatrix} 14 & 10 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 89 & 44 & 45 \\ 119 & 58 & 61 \end{bmatrix} =$$

$$\begin{bmatrix} 14(89)+10(119) & 14(44)+10(58) & 14(45)+10(61) \\ 7(89)+8(119) & 7(44)+8(58) & 7(45)+8(61) \end{bmatrix}$$

Revenue Cost Profit

$$= \begin{bmatrix} 2436 & \mathbf{1196} & 1240 \\ 1575 & \mathbf{772} & 803 \end{bmatrix} \begin{matrix} \text{Store 1} \\ \text{Store 2} \end{matrix}$$

The total cost for skateboards for store 1 is $1196 and for store 2 is $772.

CHECK IT OUT! **3.** Change store 2's inventory to 6 complete and 9 super complete. Update the product matrix, and find the profit for store 2.

A **square matrix** is any matrix that has the same number of rows as columns; it is an $n \times n$ matrix. The **main diagonal** of a square matrix is the diagonal from the upper left corner to the lower right corner.

The **multiplicative identity matrix** is any square matrix, named with the letter I, that has all of the entries along the main diagonal equal to 1 and all of the other entries equal to 0.

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix I is the multiplicative identity when A is any square matrix and $AI = IA = A$.

For $A = \begin{bmatrix} 5 & 7 \\ -1 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$$AI = \begin{bmatrix} 5 & 7 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5(1)+7(0) & 5(0)+7(1) \\ -1(1)+4(0) & -1(0)+4(1) \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ -1 & 4 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1(5)+0(-1) & 1(7)+0(4) \\ 0(5)+1(-1) & 0(7)+1(4) \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ -1 & 4 \end{bmatrix} = A$$

Because square matrices can be multiplied by themselves any number of times, you can find powers of square matrices.

EXAMPLE 4 Finding Powers of Square Matrices

$$A = \begin{bmatrix} 7 & 3 \\ -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 1 \\ 5 & 0 & -2 \\ 1 & -1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & -2 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

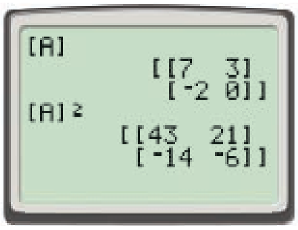
Evaluate, if possible.

A A^2

$$A^2 = \begin{bmatrix} 7 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7(7) + 3(-2) & 7(3) + 3(0) \\ -2(7) + 0(-2) & -2(3) + 0(0) \end{bmatrix}$$

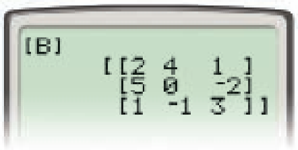
$$= \begin{bmatrix} 43 & 21 \\ -14 & -6 \end{bmatrix}$$

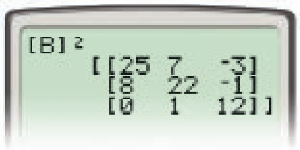


Check Use a calculator.

B B^2

For large matrices, use a graphing calculator.





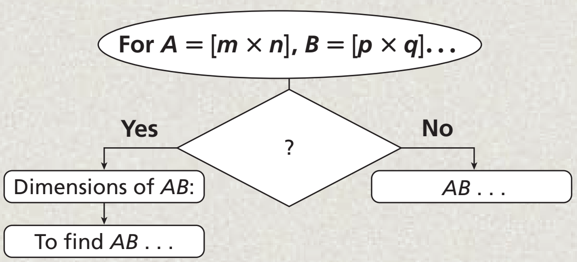
CHECK IT OUT! **Evaluate, if possible.**

4a. C^2 **4b.** A^3 **4c.** B^3 **4d.** I^4

THINK AND DISCUSS

1. Describe what happens when you try to find the first element of AB if both A and B have dimensions 2×3.
2. Tell whether matrix multiplication is commutative.
3. A is a 4×2 matrix. Can you find A^2? Why or why not?
4. **GET ORGANIZED** Copy and complete the graphic organizer. In the decision diamond, enter a question to determine whether AB is defined. Then give the general procedure for finding AB, if it is defined.

Know it! Note



4-2 Exercises

California Standards
Preparation for 🔑 **2.0**; Review of **7MG3.4**

go.hrw.com
Homework Help Online
KEYWORD: MB7 4-2
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

1. Vocabulary A 2 × 2 matrix with every entry equal to 1 is a __?__. (*square matrix* or *multiplicative identity matrix*)

SEE EXAMPLE 1 p. 253

Tell whether each product is defined. If so, give its dimensions.

2. $A_{4\times 5}$ and $B_{5\times 3}$; AB **3.** $A_{4\times 5}$ and $B_{5\times 3}$; BA **4.** $C_{9\times 5}$ and $D_{5\times 9}$; CD

5. $C_{9\times 5}$ and $D_{5\times 9}$; DC **6.** $E_{6\times 2}$ and $F_{2\times 6}$; EF **7.** $E_{6\times 2}$ and $F_{2\times 6}$; FE

SEE EXAMPLE 2 p. 254

Use the following matrices for Exercises 8–13. Find each product, if possible.

$$A=\begin{bmatrix}0&7&3\\-2&3&0\end{bmatrix}\quad B=\begin{bmatrix}4&2\\1&-3\end{bmatrix}\quad C=\begin{bmatrix}-3&1\\5&-2\\0&1\end{bmatrix}\quad D=\begin{bmatrix}3&-1&7&10\\1&-1&3&5\end{bmatrix}\quad I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

8. BA **9.** CA **10.** CB

11. DC **12.** BI **13.** IB

SEE EXAMPLE 3 p. 255

14. Recycling Students collected recyclables for fund-raising over a three-week period. Use matrix multiplication to find the total amount of money collected for each type of item.

Recyclables Collected (lb)			
Item	Week 1	Week 2	Week 3
Glass	29	25	16
Cans	8	11	6
Newspaper	163	127	206
Office paper	53	107	84

Price Per Pound ($)				
Week	Glass	Cans	News-paper	Office Paper
1	0.02	0.70	0.02	1.06
2	0.02	0.55	0.01	1.00
3	0.01	0.42	0.02	1.03

SEE EXAMPLE 4 p. 256

Use the following matrices for Exercises 15–18. Evaluate, if possible.

$$A=\begin{bmatrix}-1&-2\\1&0\end{bmatrix}\quad B=\begin{bmatrix}3&4&2\\-1&0&0\\3&0&1\end{bmatrix}\quad C=\begin{bmatrix}3&1\\0&-2\\1&1\end{bmatrix}$$

15. A^2 **16.** A^3 **17.** C^2 **18.** B^2

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
19–24	1
25–29	2
30	3
31–40	4

Extra Practice
Skills Practice p. S10
Application Practice p. S35

Tell whether each product is defined. If so, give its dimensions.

19. $A_{2\times 1}$ and $B_{2\times 3}$; AB **20.** $A_{2\times 1}$ and $B_{2\times 3}$; BA **21.** $C_{3\times 5}$ and $D_{5\times 1}$; CD

22. $C_{3\times 5}$ and $D_{5\times 1}$; DC **23.** $E_{7\times 7}$ and $F_{6\times 7}$; EF **24.** $E_{7\times 7}$ and $F_{6\times 7}$; FE

Use the following matrices for Exercises 25–29. Find each product, if possible.

$$A=\begin{bmatrix}4\\-1\\2\end{bmatrix}\quad B=\begin{bmatrix}-3&0\\7&-2\\0&1\end{bmatrix}\quad C=\begin{bmatrix}-2&3&-4\\1&-1&1\\4&1&3\end{bmatrix}\quad I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

25. AB **26.** CA **27.** CB **28.** IC **29.** CI

30. **Inventory** A pet stroller comes in two sizes. Two stores have inventories as shown in the first table. Find the total cost of the pet strollers for each store.

Pet Stroller Inventory		
	Standard	Large
Store 1	11	7
Store 2	8	6

Pet Stroller Profits			
	Revenue ($)	Store Cost ($)	Profit ($)
Standard	130	75	55
Large	190	110	80

Use the following matrices for Exercises 31–40. Simplify, if possible.

$$Q = \begin{bmatrix} 4 & 13 & -9 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 4 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 5 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

31. S^2
32. B^2
33. T^2
34. S^3
35. Q^3
36. AB
37. BA
38. $2BA - C$
39. $3CB + 2B$
40. $(BA)^2$

Diving In a diving competition, the point total for each dive is multiplied by an assigned degree of difficulty to determine the diver's score.

Points for Each Dive			
Diver	Dive 1	Dive 2	Dive 3
Ted	23.0	18.5	19.5
Chloe	24.0	28.5	25.0
Biko	19.0	22.0	21.5
Hana	27.0	26.5	28.0

Degree of Difficulty Multiplier				
Dive	Ted	Chloe	Biko	Hana
1	1.2	1.6	2.0	1.8
2	2.3	2.0	2.8	2.5
3	2.7	2.6	3.2	3.1

a. Organize the tables as matrices, and multiply.
b. Use the product matrix to find the scores for each of the four divers.
c. Explain why only the numbers on the main diagonal of the product matrix are meaningful in the context of the problem.

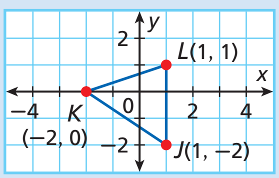
Critical Thinking **For Exercises 42–45, tell whether each statement is always, sometimes, or never true for matrices *A* and *B*. Explain your answer.**

42. If A is 2×3 and B has three rows, then AB is defined.
43. If A is 2×3 and B has three columns, then AB is defined.
44. If AB is defined, then BA is defined.
45. If both AB and BA are defined, both are square matrices.

CONCEPT CONNECTION

46. This problem will prepare you for the Concept Connection on page 268.

a. Place the vertices of the triangle in a matrix so that the x-coordinates are in row 1 and the y-coordinates are in row 2.



b. Use the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ to multiply each x- and y-coordinate by 2.

c. Draw a new triangle using the new coordinates. Describe the new triangle.

47. Solve for x: $\begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 6 & \frac{x}{2} \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 21 & -19 \\ 24 & -26 \end{bmatrix}$

48. Write About It Explain why $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

49. Fiber Arts The first table shows points awarded by the judges at the New England Sheep & Wool Fair for each competition. The second table shows the multiplier used for the degree of difficulty of each piece. Find the total score for each contestant.

Points Awarded			
Contestant	Wall Hanging	Clothing	Rug
Madison	16.5	18.0	17.5
Devyn	12.5	14.0	17.0
Ali	16.0	19.5	18.0

Degree of Difficulty Multiplier			
Category	Madison	Devyn	Ali
Wall Hanging	2	3	2
Clothing	3	3	1
Rug	2	2	1

50. Sales Old and new commission rates for shoe sales are given.

a. Find the product matrix. How much did each person make under each rate?

b. Which salesperson benefited the most from the change in rates? Explain.

Total Sales ($)			
Salesperson	Men's	Women's	Children's
Leigh	5200	4200	2300
Khalid	8100	8400	3100
Ari	2700	7400	630

Commission Rates		
Shoe	Old Rate	New Rate
Men's	9%	9.5%
Women's	9%	10%
Children's	13%	12%

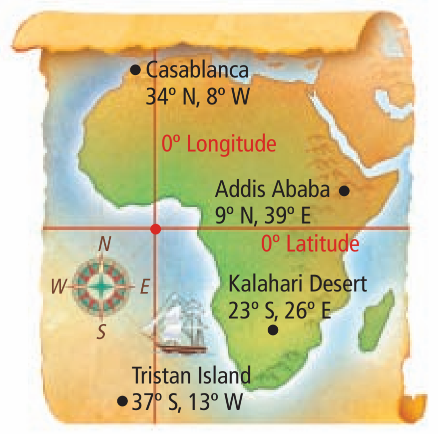
51. Puzzle Contestants in a reality TV show need to get to a location given by entries in the following matrix product:

$$P = \begin{bmatrix} 5 & 1 \\ -11 & 2 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 9 & -3 \end{bmatrix}$$

latitude: p_{21} (north if positive, south if negative)

longitude: p_{12} (east if positive, west if negative)

What is the location that the contestants must make their way to?



52. Football Find the total number of points scored by each team.

Team	Touchdowns	Extra Points	Field Goals
Redcliffe	11	9	4
Mayson	15	12	6
Rye Harbor	6	5	9

Type of Score	Points
Touchdown	6
Extra point	1
Field goal	3

53. Critical Thinking Write A as a scalar product where each entry is a whole number.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{3}{4} & \frac{5}{6} \end{bmatrix}$$

STANDARDIZED TEST PREP

54. B is a 5×12 matrix. For AB to be defined, what characteristic must A have?

Ⓐ 5 columns Ⓑ 12 columns Ⓒ 5 rows Ⓓ 12 rows

55. Which result is NOT equal to the other three?

Ⓕ $2\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Ⓖ $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Ⓗ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Ⓙ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

56. For the matrix product $P = \begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 3 & 8 \end{bmatrix}$, which expression gives the value of p_{22}?

Ⓐ $4(-2) + 2(3)$ Ⓑ $7(5) + (-1)8$ Ⓒ $4(5) + 2(8)$ Ⓓ $(-1)3 + 2(8)$

57. **Short Response** For $A = \begin{bmatrix} 3 & 4 \\ -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 \\ 6 & 8 \end{bmatrix}$, tell whether AB, BA, or neither equals $\begin{bmatrix} 33 & -18 \\ -14 & 64 \end{bmatrix}$.

CHALLENGE AND EXTEND

58. Is matrix multiplication associative? That is, does $ABC = (AB)C = A(BC)$ if the products are defined? Give an example to support your answer.

59. To write the *transpose* A^T of a matrix A for $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$, reverse its rows and columns.

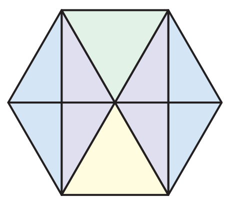
a. Can a matrix always be multiplied by its transpose? Explain.

b. Find $P = AA^T$ for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Which entries of the product are equal?

60. On a calculator, enter matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Multiply A by itself, and record the value of the entry in row 2 column 2 of the product matrix. Continue to multiply by A and record the entry in this location. What is the relationship between successive recorded values?

SPIRAL REVIEW

Graphic Design **The outer shape of this design is a regular hexagon. The green triangle is an equilateral triangle.** *(Previous course)*



61. How many pairs of vertical angles are in the design?

62. How many triangles are congruent to the green triangle?

63. How many line segments are congruent to one side of the hexagon?

Graph each point in three-dimensional space. *(Lesson 3-5)*

64. $(0, 4, -5)$ **65.** $(2, 2, 6)$ **66.** $(-3, -3, 3)$ **67.** $(1, -1, -1)$

Evaluate, if possible. *(Lesson 4-1)*

$$S = \begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 0.5 & 0.83 \\ 5 & 0 \end{bmatrix} \quad V = \begin{bmatrix} 2 & 3 & 0 \\ -4 & 1 & -1 \end{bmatrix}$$

68. $S + T$ **69.** $V - T$ **70.** $4T$

Connecting Algebra to **Geometry**

See Skills Bank page S63

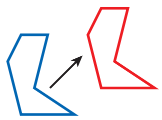
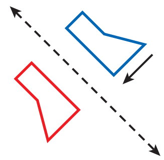
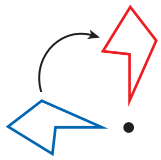
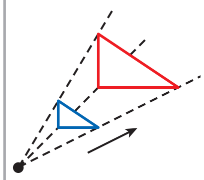
Transformations

California Standards

Review of 7MG3.2 Understand and use coordinate graphs to plot simple figures, determine lengths and areas related to them, **and determine their image under translations and reflections.**

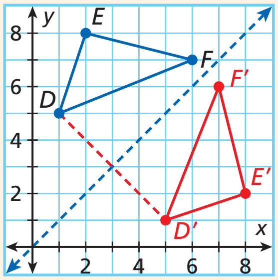
A *transformation* describes a way of moving or resizing a geometric figure. *Rigid transformations*, or *isometries*, do not change the size and shape of figures. However, not all transformations are rigid.

Transformations are described by distances, angle measures, and lines of reflection, depending on the type of transformation. The properties of a transformation tell you what attributes of the figure remain unchanged.

	Translation	Reflection	Rotation	Dilation
What You Need to Describe It	Horizontal and vertical distance	Line of reflection	Center angle of rotation	Center scale factor
What Does Not Change	Size and shape, area, orientation	Size and shape, area	Size and shape, area, orientation	Orientation
				

Example

Reflect $\triangle DEF$ across the line $y = x$.



Step 1 Draw a line through D perpendicular to the line of reflection. Mark point D' as the image of point D. Point D and point D' must be the same distance from the line of reflection.

Step 2 Repeat Step 1 for points E and F. Connect the points to make $\triangle D'E'F'$.

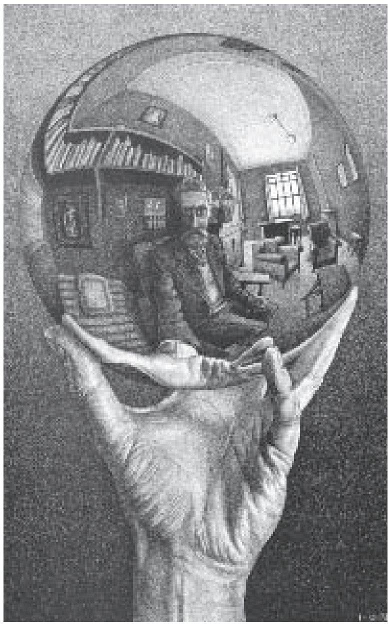
Try This

Use graph paper to show each transformation.

1. Plot rectangle $PQRS$ with vertices $P(3, 1)$, $Q(3, -2)$, $R(-2, -2)$, and $S(-2, 1)$. Rotate the rectangle 90° clockwise. Use vertex P as the center of rotation.
2. Plot $\triangle ABC$ with vertices $A(1, 4)$, $B(6, 4)$, and $C(4, 6)$. Enlarge the triangle using the origin as the center of dilation with a scale factor of 1.5.
3. The identity transformation I maps each point of the plane onto itself. Describe consecutive reflections that are equivalent to the identity transformation.
4. Draw the horizontal line $y = 4$. Use $\triangle DEF$ from the example. Translate the triangle 3 units to the right, and then reflect it across the line. Repeat twice. This is an example of a *glide reflection*, the product of a reflection in a line and a translation along the same line.

4-3

Using Matrices to Transform Geometric Figures


The Granger Collection, New York

Objective
Use matrices to transform a plane figure.

Vocabulary
translation matrix
reflection matrix
rotation matrix

Who uses this?
Artists, such as M. C. Escher, may use repeated transformed patterns to create their work. (See Exercise 16.)

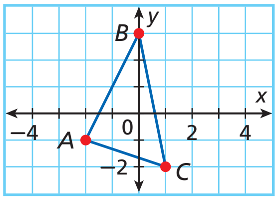
You can describe the position, shape, and size of a polygon on a coordinate plane by naming the ordered pairs that define its vertices.

The coordinates of $\triangle ABC$ below are $A(-2, -1)$, $B(0, 3)$, and $C(1, -2)$.

You can also define $\triangle ABC$ by a matrix:

$$P = \begin{bmatrix} -2 & 0 & 1 \\ -1 & 3 & -2 \end{bmatrix} \begin{matrix} \leftarrow x\text{-coordinates} \\ \leftarrow y\text{-coordinates} \end{matrix}$$

A **translation matrix** is a matrix used to translate coordinates on the coordinate plane. The matrix sum of a *preimage* and a translation matrix gives the coordinates of the translated *image*.



EXAMPLE 1

Using Matrices to Translate a Figure

Translate $\triangle ABC$ with coordinates $A(-2, -1)$, $B(0, 3)$, and $C(1, -2)$ 2 units right and 3 units down. Find the coordinates of the vertices of the image, and graph.

The translation matrix will have 2 in all entries in row 1 and −3 in all entries in row 2.

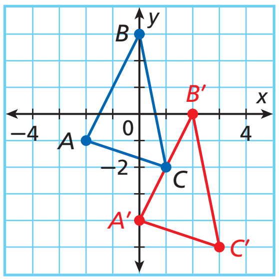
$$\begin{bmatrix} 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix} \begin{matrix} \leftarrow x\text{-translation} \\ \leftarrow y\text{-translation} \end{matrix}$$

Coordinate matrix + *Translation matrix*

$$\begin{bmatrix} -2 & 0 & 1 \\ -1 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -2+2 & 0+2 & 1+2 \\ -1-3 & 3-3 & -2-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 3 \\ -4 & 0 & -5 \end{bmatrix}$$

$A'B'C'$, the image of $\triangle ABC$, has coordinates $A'(0, -4)$, $B'(2, 0)$, and $C'(3, -5)$.



Reading Math
The prefix *pre-* means "before," so the *preimage* is the original figure before any transformations are applied. The *image* is the resulting figure after a transformation.

CHECK IT OUT!
1. Translate $\triangle GHJ$ with coordinates $G(2, 4)$, $H(3, 1)$, and $J(1, -1)$ 3 units right and 1 unit down. Find the coordinates of the vertices of the image and graph.

A *dilation* is a transformation that scales—enlarges or reduces—the preimage, resulting in similar figures. Remember that for similar figures, the shape is the same but the size may be different. Angles are congruent, and side lengths are proportional.

When the *center of dilation* is the origin, multiplying the coordinate matrix by a scalar gives the coordinates of the dilated image. In this lesson, all dilations assume that the origin is the center of dilation.

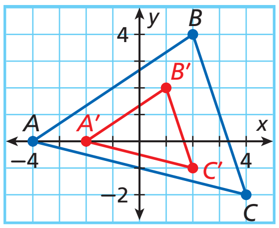
EXAMPLE 2 Using Matrices to Dilate a Figure

Reduce triangle $\triangle ABC$ with coordinates $A(-4, 0)$, $B(2, 4)$, and $C(4, -2)$ by a factor of $\frac{1}{2}$. Find the coordinates of the vertices of the image, and graph.

Multiply each coordinate by $\frac{1}{2}$ by multiplying each entry by $\frac{1}{2}$.

$$\frac{1}{2}\begin{bmatrix} -4 & 2 & 4 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(-4) & \frac{1}{2}(2) & \frac{1}{2}(4) \\ \frac{1}{2}(0) & \frac{1}{2}(4) & \frac{1}{2}(-2) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{matrix} \leftarrow x\text{-coordinates} \\ \leftarrow y\text{-coordinates} \end{matrix}$$

$A'B'C'$, the image of $\triangle ABC$, has coordinates $A'(-2, 0)$, $B'(1, 2)$, and $C'(2, -1)$.



CHECK IT OUT! 2. Enlarge $\triangle DEF$ with coordinates $D(2, 3)$, $E(5, 1)$, and $F(-2, -7)$ a factor of $\frac{4}{3}$. Find the coordinates of the vertices of the image, and graph.

A **reflection matrix** is a matrix that creates a mirror image by reflecting each vertex over a specified line of symmetry. To reflect a figure across the y-axis, multiply $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ by the coordinate matrix. This **reverses the x-coordinates** and **keeps the y-coordinates unchanged.**

EXAMPLE 3 Using Matrices to Reflect a Figure

Reflect $\triangle JKL$ with coordinates $J(3, 4)$, $K(4, 2)$, and $L(1, -2)$ across the y-axis. Find the coordinates of the vertices of the image, and graph.

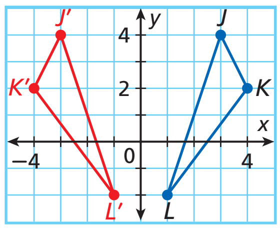
Caution!
Matrix multiplication is not commutative. So be sure to keep the transformation matrix on the left!

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 4 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -3 & -4 & -1 \\ 4 & 2 & -2 \end{bmatrix}$$

Each x-coordinate is multiplied by -1.

Each y-coordinate is multiplied by 1.

The coordinates of the vertices of the image are $J'(-3, 4)$, $K'(-4, 2)$, and $L'(-1, -2)$.



CHECK IT OUT! 3. To reflect a figure across the x-axis, multiply by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Reflect $\triangle JKL$ across the x-axis. Find the coordinates of the vertices of the image and graph.

A **rotation matrix** is a matrix used to rotate a figure. Example 4 gives several types of rotation matrices.

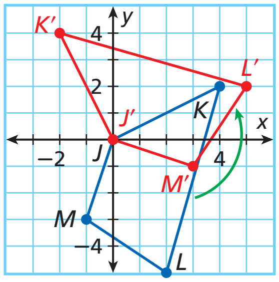
EXAMPLE 4 Using Matrices to Rotate a Figure

Use each matrix to rotate polygon $JKLM$ with coordinates $J(0, 0)$, $K(4, 2)$, $L(2, -5)$, and $M(-1, -3)$ about the origin. Graph and describe the image.

A $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 & 2 & -1 \\ 0 & 2 & -5 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 5 & 3 \\ 0 & 4 & 2 & -1 \end{bmatrix}$$

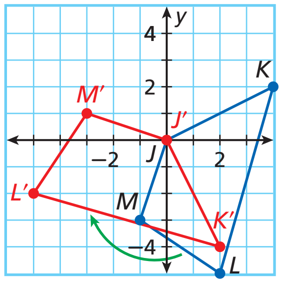
The image is rotated 90° counterclockwise.



B $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 & 2 & -1 \\ 0 & 2 & -5 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -5 & -3 \\ 0 & -4 & -2 & 1 \end{bmatrix}$$

The image is rotated 90° clockwise.



Helpful Hint

Multiplying a coordinate by −1 results in the opposite of the coordinate.

CHECK IT OUT!

4. Use $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Rotate $\triangle ABC$ with coordinates $A(0, 0)$, $B(4, 0)$, and $C(0, -3)$ about the origin. Graph and describe the image.

THINK AND DISCUSS

1. Describe the transformation resulting from multiplying a coordinate matrix by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2. Describe what happens to an x-coordinate in a matrix when multiplied by this row of a transformation matrix.

a. $\begin{bmatrix} 1 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 1 \end{bmatrix}$ **c.** $\begin{bmatrix} 0.5 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 1 \end{bmatrix}$

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer. Q is a triangle represented by its 2×3 coordinate matrix. Complete the summary by filling in a matrix expression.

Transformation	Matrix Operation
Translate Q vertically	
Translate Q horizontally	
Enlarge or reduce Q.	
Reflect Q across the x-axis or y-axis	
Rotate Q 90° clockwise or counterclockwise.	

4-3 Exercises

go.hrw.com
Homework Help Online
KEYWORD: MB7 4-3
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

1. **Vocabulary** A __?__ creates a mirror image of a set of points. (*reflection matrix* or *translation matrix*)

SEE EXAMPLE 1 p. 262

Translate the polygon with coordinates $P(-2, 4)$, $Q(3, 1)$, $R(1, -4)$, and $S(-2, -2)$ as indicated. Find the coordinates of the vertices of the image, and graph.

2. 2 units left and 1 unit up
3. 1 unit right and 0 units down

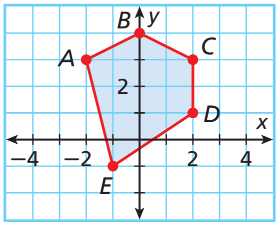
SEE EXAMPLE 2 p. 263

Use a matrix to reduce or enlarge the polygon with coordinates $P(-2, 4)$, $Q(3, 1)$, $R(1, -4)$, and $S(-2, -2)$ by the given factor. Find the coordinates of the vertices of the image, and graph.

4. Reduce polygon *PQRS* by a factor of 0.5.
5. Enlarge polygon *PQRS* by a factor of 2.

SEE EXAMPLE 3 p. 263

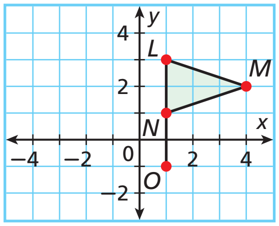
Reflect the figure with coordinates $A(-2, 3)$, $B(0, 4)$, $C(2, 3)$, $D(2, 1)$, and $E(-1, -1)$ across the given line. Find the coordinates of the vertices of the image, and graph.



6. Reflect *ABCDE* across the *y*-axis.
7. Use $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to reflect *ABCDE* across the line $y = x$.

SEE EXAMPLE 4 p. 264

Use each matrix to rotate the figure with coordinates $L(1, 3)$, $M(4, 2)$, $N(1, 1)$, and $O(1, -1)$ about the origin. Graph and describe the image.



8. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
9. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
10	1
11	2
12	3
13–14	4

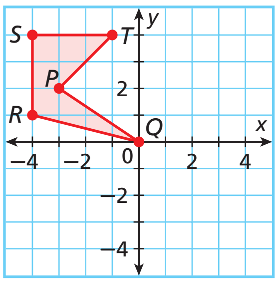
Extra Practice
Skills Practice p. S10
Application Practice p. S35

10. Translate the polygon with coordinates $D(0, 4)$, $E(-3, -1)$, $F(1, -5)$, and $G(1, 0)$ 3 units right and 3 units up. Find the coordinates of the vertices of the image, and graph.

11. Dilate the polygon with coordinates $W(1, 2)$, $X(-2, 3)$, $Y(-3, 4)$, and $Z(-4, 1)$ by a factor of $\frac{3}{2}$. Find the coordinates of the vertices of the image, and graph.

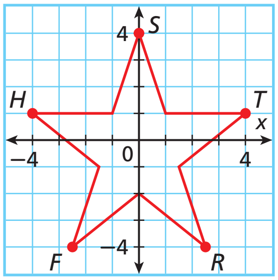
12. Reflect the figure with coordinates $A(-2, 3)$, $B(0, 4)$, $C(2, 3)$, $D(2, 1)$, and $E(-1, -1)$ across the *x*-axis. Graph and describe the image.

Use each matrix to rotate the figure *PQRST* with coordinates $P(-3, 2)$, $Q(0, 0)$, $R(-4, 1)$, $S(-4, 4)$, and $T(-1, 4)$. Graph and describe the image.



13. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
14. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

15. Design Skye creates a design based on a starfish as a background for her school ecology club Web page. On a coordinate plane, the ends of the arms of the first image are $S(0, 4)$, $T(4, 1)$, $R(2.5, -4)$, $F(-2.5, -4)$, and $H(-4, 1)$.



a. Use the matrix $\begin{bmatrix} 0.81 & -0.59 \\ 0.59 & 0.81 \end{bmatrix}$ to rotate the star through $\frac{1}{10}$ of a circle. Round the coordinates of the new image to the nearest half-unit.

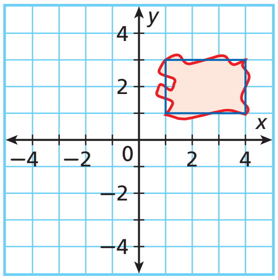
b. Does the star rotate clockwise or counterclockwise? Explain.

LINK

Art

Artist M. C. Escher (1898–1972) transformed symmetric geometric shapes into birds, reptiles, and other figures.

16. Art To make a *tessellation,* which is a picture made entirely of repeated transformations of figures without gaps or overlaps, an artist creates the initial figure and transforms it repeatedly.



a. The artist first rotates the figure 180°. Write a rotation matrix for this transformation.

b. Find the vertices of the figure after the rotation matrix is applied.

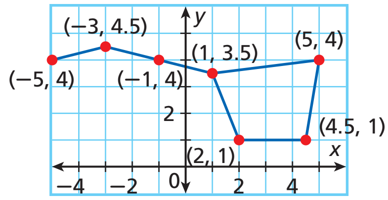
c. Next, the artist translates the figure 4 units up and 2 units right. Write a translation matrix for this transformation.

d. Find the vertices of the figure after this second transformation is applied.

e. Sketch the original figure and the transformed figure on the same coordinate grid.

17. Critical Thinking $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Explain what happens if you multiply T by the coordinate matrix of a figure and then multiply T by the result.

Use a matrix to perform each transformation on the graph representing the constellation the Big Dipper. Find the coordinates of the image.



18. translation 2 units up

19. translation 1 unit down and 3 units left

20. enlargement by a factor of 2

21. reflection across the x-axis

22. rotation 90° clockwise

23. rotation 90° counterclockwise

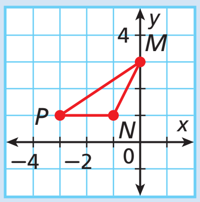
24. Write About It What does multiplying $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by a coordinate matrix do to the figure on the coordinate plane?

25. What transformation matrix represents $g(x) = -f(x)$? What transformation represents $g(x) = f(-x)$?

CONCEPT CONNECTION

26. This problem will prepare you for the Concept Connection on page 268.

a. Place the vertices of the triangle in a coordinate matrix.



b. Multiply the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ by the coordinate matrix.

c. Draw a new triangle using the new coordinates. Describe the image.

d. Repeat parts **b** and **c** using the new triangle as the preimage. Describe the final triangle.

STANDARDIZED TEST PREP

27. To create a quilt pattern, Morgan dilates a figure, rotates it 90° clockwise, and reflects it across the *y*-axis. Which sequence results in the image?

Ⓐ scalar multiplication; matrix addition; matrix multiplication

Ⓑ scalar multiplication; matrix multiplication; matrix multiplication

Ⓒ matrix addition; matrix multiplication; matrix addition

Ⓓ matrix multiplication; matrix addition; scalar multiplication

28. What effect does multiplying the coordinates of a figure by $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ have?

Ⓕ The figure is enlarged and rotated 90° clockwise.

Ⓖ The figure is reduced and rotated 90° counterclockwise.

Ⓗ The figure is reduced and reflected across the *x*-axis.

Ⓙ The figure is enlarged and reflected across the *y*-axis.

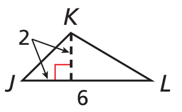
29. Which matrix can be used to rotate a figure 180° about the origin?

Ⓐ $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ Ⓒ $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ Ⓓ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

CHALLENGE AND EXTEND

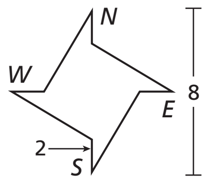
30. What matrix could you use to reflect a figure across the line $y = -x$?

31. Position $\triangle JKL$ on a coordinate plane, and assign coordinates to the vertices.



a. How can you transform $\triangle JKL$ to create a symmetrical compass with four points *N*, *E*, *S*, and *W*?

b. Use matrices to transform $\triangle JKL$, and give the coordinates of the four points *N*, *E*, *S*, and *W*.



32. Transform a figure using $\begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$. Describe the transformation. What would happen if this transformation were performed repeatedly?

SPIRAL REVIEW

33. Determine whether the data set could represent a linear function. *(Lesson 2-3)*

Tickets	2	5	8	11
Cost ($)	35.00	87.50	140.00	192.50

Determine if the given point is a solution of the system of inequalities. *(Lesson 3-3)*

34. $(2, -4)\ \begin{cases} y > 2x - 8 \\ y \le \frac{1}{4}x + 2 \end{cases}$

35. $(0, 5)\ \begin{cases} y > 0 \\ y \ge 2x - 11 \\ 5x + y < 5.5 \end{cases}$

Evaluate, if possible. *(Lesson 4-2)*

36. $\begin{bmatrix} 5 & -5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 10 & 1 \\ -2 & 0 \end{bmatrix}$

37. $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

38. $\begin{bmatrix} 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

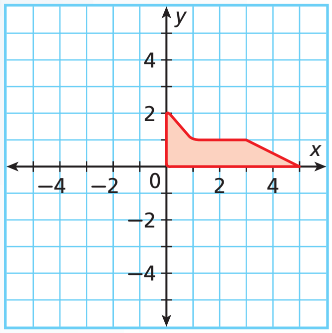
CHAPTER 4

SECTION 4A

Concept Connection

Games Away

You are making a video game using the space shuttle figure shown on the grid. By applying different transformations to the shuttle, you can simulate different moves.



1. Find the coordinates of the vertices of the shuttle. Write the coordinates in matrix form.
2. The *hyperjump* button causes the shuttle to immediately rise 4 units. What transformation represents the shuttle after hyperjump? Write the transformation matrix, and show the matrix operation.
3. What transformation will make the shuttle reverse direction? Write the transformation matrix, and show the matrix operation.
4. What transformation will make the shuttle fly upside down? Write the transformation matrix, and show the matrix operation.
5. What matrix will rotate the shuttle 180° about the origin? Show the matrix operation.
6. Suppose you reflect the shuttle over one axis and then the other. Compare the result to the rotation in Problem 5.
7. If the shuttle is hit by an asteroid, it is reduced by a factor of $\frac{1}{2}$. What matrix operation will show the reduction? What is the ratio of the area of the preimage to that of the image?

READY TO GO ON?

Quiz for Lessons 4-1 Through 4-3

4-1 Matrices and Data

Use the table for Problems 1–4.

1. Display the data in the form of a matrix M.
2. What are the dimensions of M?
3. What is the value of the matrix entry with the address m_{32}? What does it represent?
4. What is the address of the entry that has the value 90?

Olympic Medal Specifications

	Gold	Silver	Bronze
Weight (lb)	1.25	1.25	1
% copper	7.5	7.5	90
Hours of handicrafting	19.65	18.30	18.45

Use the matrices below for Problems 5–8. Evaluate, if possible.

$$A = \begin{bmatrix} 3 & 4 \\ 1 & -2 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 1 & -1 \\ -1.5 & 2 & -2 \end{bmatrix}$$

5. $A + C$
6. $2B$
7. $C - D$
8. $C - 3A$

4-2 Multiplying Matrices

Use the matrices named below for Problems 9–12. Tell whether each product is defined. If so, give its dimensions.

$P_{5\times2}$, $Q_{2\times5}$, $R_{1\times5}$, and $S_{5\times2}$

9. PQ
10. QR
11. RS
12. SP

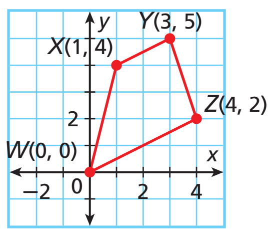
Use the matrices below for Problems 13–16. Evaluate, if possible.

$$E = \begin{bmatrix} 1 & -2 & -1 \\ 5 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix} \quad F = \begin{bmatrix} 0.5 & 0.75 & -1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \quad H = \begin{bmatrix} -1 & 4 \\ 2 & 0 \\ 0 & -1 \end{bmatrix}$$

13. EF
14. FH
15. HG
16. G^2

4-3 Using Matrices to Transform Geometric Figures

For Problems 17–20, use polygon $WXYZ$ with coordinates $W(0, 0)$, $X(1, 4)$, $Y(3, 5)$, and $Z(4, 2)$. Give the coordinates of the image and graph.



17. Translate polygon $WXYZ$ 1 unit to the left and 2 units down.
18. Reduce polygon $WXYZ$ by a factor of $\frac{2}{3}$.
19. Use $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to transform polygon $WXYZ$. Describe the image.
20. Use $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ to transform polygon $WXYZ$. Describe the image.
21. How does multiplying by $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ transform polygon $WXYZ$?

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

4-4 Determinants and Cramer's Rule

Objectives

Find the determinants of 2×2 and 3×3 matrices.

Use Cramer's rule to solve systems of linear equations.

Vocabulary

determinant
coefficient matrix
Cramer's rule

Who uses this?

Sports nutritionists planning menus need to solve systems of equations for Calories and grams of protein, fat, and carbohydrates. (See Example 4.)

Every square matrix (n by n) has an associated value called its *determinant,* shown by straight vertical brackets, such as $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$. The determinant is a useful measure, as you will see later in this lesson.

Know it! Note

Determinant of a 2 × 2 Matrix

WORDS	NUMBERS	ALGEBRA
The **determinant** of a 2 by 2 matrix is the difference of the products of the diagonals	$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$ $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (3)(2) = -2$	$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$

Reading Math

The determinant of matrix A may be denoted as det A or $|A|$. Don't confuse the $|A|$ notation with absolute value notation.

EXAMPLE 1 Finding the Determinant of a 2 × 2 Matrix

Find the determinant of each matrix.

A $\begin{bmatrix} 6 & 5 \\ 8 & 3 \end{bmatrix}$

$\begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix} = 6(3) - 8(5)$ *Find the difference of the cross products.*

$= 18 - 40 = -22$

The determinant is -22.

B $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -6 & 3 \end{bmatrix}$

$\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -6 & 3 \end{vmatrix} = \frac{1}{3}(3) - (-6)\left(\frac{2}{3}\right) = 1 + 4 = 5$

The determinant is 5.

California Standards

2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or **with matrices.**

CHECK IT OUT! **Find the determinant of each matrix.**

1a. $\begin{bmatrix} 0.2 & 30 \\ -0.3 & 5 \end{bmatrix}$ **1b.** $\begin{bmatrix} \frac{1}{3} & 3 \\ \frac{5}{6} & \frac{3}{4} \end{bmatrix}$ **1c.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 4 & 2\pi \end{bmatrix}$

You can use the determinant of a matrix to help you solve a system of equations. For two equations with two variables written in $ax + by = c$ form, you can construct a matrix of the coefficients of the variables.

For the system $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$, the coefficient matrix is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

The **coefficient matrix** for a system of linear equations in standard form is the matrix formed by the coefficients for the variables in the equations.

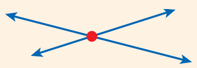
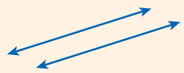
The determinant D of the coefficient matrix is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Know it! Note

Cramer's Rule for Two Equations

$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ has solutions $x = \dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D}$, $y = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D}$, where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

You can use Cramer's rule to tell whether the system represented by the matrix has one solution, no solution, or infinitely many solutions.

Solutions of Systems		
If $D \neq 0$, the system is consistent and has **one** unique solution.	If $D = 0$ and *at least one* numerator determinant is 0, the system is dependent and has **infinitely many** solutions.	If $D = 0$ and *neither* numerator determinant is 0, the system is inconsistent and has no solution.
		

EXAMPLE 2

Using Cramer's Rule for Two Equations

Use Cramer's rule to solve each system of equations.

A $\begin{cases} x - y = 3 \\ 2x - y = -1 \end{cases}$

Step 1 Find D, the determinant of the coefficient matrix. $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$D = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1(-1) - 2(-1) = 1$ *$D \neq 0$, so the system is consistent.*

Step 2 Solve for each variable by replacing the coefficients of that variable with the constants as shown below.

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix}}{1} = -4$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_1 & c_2 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}}{1} = -7$$

The solution is $(-4, -7)$.

Use Cramer's rule to solve each system of equations.

B $\begin{cases} y - 2 = 3x \\ 3x - y = 7 \end{cases}$

Step 1 Write the equations in standard form. $\begin{cases} 3x - y = -2 \\ 3x - y = 7 \end{cases}$

Step 2 Find the determinant of the coefficient matrix.

$$D = \begin{vmatrix} 3 & -1 \\ 3 & -1 \end{vmatrix} = -3 - (-3) = 0$$

$D = 0$, so the system is either inconsistent or dependent. Check the numerators for x and y to see if either is 0.

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{0} \rightarrow \begin{vmatrix} -2 & -1 \\ 7 & -1 \end{vmatrix} = 9 \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{0} \rightarrow \begin{vmatrix} 3 & -2 \\ 3 & 7 \end{vmatrix} = 27$$

Neither numerator is 0. The system is inconsistent with no solutions.

CHECK IT OUT! **2.** Use Cramer's rule to solve. $\begin{cases} 6x - 2y = 14 \\ 3x = y + 7 \end{cases}$

To apply Cramer's rule to 3×3 systems, you need to find the determinant of a 3×3 matrix. One method is shown below.

Rewrite the first two columns at the right side of the determinant.

Add the sum of the products of the red diagonals. Then subtract the sum of the blue diagonals.

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \left| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \right| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} \quad a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

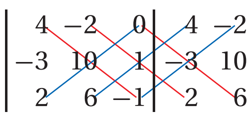
EXAMPLE 3 Finding the Determinant of a 3 × 3 Matrix

Find the determinant of ***A*****.**

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -3 & 10 & 1 \\ 2 & 6 & -1 \end{bmatrix} \quad \det A = \begin{vmatrix} 4 & -2 & 0 \\ -3 & 10 & 1 \\ 2 & 6 & -1 \end{vmatrix}, \text{ so write } \left| \begin{matrix} 4 & -2 & 0 \\ -3 & 10 & 1 \\ 2 & 6 & -1 \end{matrix} \right| \begin{matrix} 4 & -2 \\ -3 & 10 \\ 2 & 6 \end{matrix}$$

Step 1 Multiply each "down" diagonal and add.

$4(10)(-1) + (-2)(1)(2) + 0(-3)(6) = -44$



Step 2 Multiply each "up" diagonal and add.

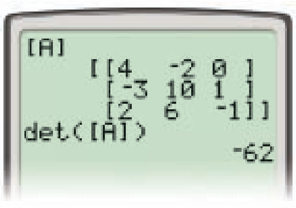
$(2)(10)(0) + (6)(1)(4) + (-1)(-3)(-2) = 18$

Step 3 Find the difference of the sums.

$-44 - 18 = -62$.

The determinant is -62.

Check Use a calculator.



Helpful Hint

Lightly draw the diagonals to help you locate the six products needed to find the determinant.

CHECK IT OUT! **3.** Find the determinant of $\begin{bmatrix} 2 & -3 & 4 \\ 5 & 1 & -2 \\ 10 & 3 & -1 \end{bmatrix}$.

Cramer's rule can be expanded to cover 3×3 systems.

Know it! Note

Cramer's Rule for Three Equations

The system $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ has solutions given by

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D},\ y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D},\ z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D} \text{ where } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D \neq 0.$$

If $D \neq 0$, then the system has a unique solution.

If $D = 0$ and no numerator is 0, then the system is inconsistent. If $D = 0$ and at least one numerator is 0, then the system may be inconsistent or dependent.

EXAMPLE 4

Nutrition Application

A nutritionist planning a diet for a football player wants him to consume 3600 Calories and 750 grams of food daily. Calories from protein and from fat will be 60% of the total Calories. How many grams of protein, carbohydrates, and fat will this diet include?

Calories per Gram

Food	Calories
Protein	4
Carbohydrates	4
Fat	9

The diet will include p grams of protein, c grams of carbohydrates, and f grams of fat.

$4p + 4c + 9f = 3600$ *Equation for total Calories*

$p + c + f = 750$ *Total grams of food*

$4p + 0c + 9f = 2160$ *Calories from protein and fat, 60%(3600) = 2160*

Use a calculator.

$$D = \begin{vmatrix} 4 & 4 & 9 \\ 1 & 1 & 1 \\ 4 & 0 & 9 \end{vmatrix} = -20 \quad p = \frac{\begin{vmatrix} 3600 & 4 & 9 \\ 750 & 1 & 1 \\ 2160 & 0 & 9 \end{vmatrix}}{D} \quad c = \frac{\begin{vmatrix} 4 & 3600 & 9 \\ 1 & 750 & 1 \\ 4 & 2160 & 9 \end{vmatrix}}{D} \quad f = \frac{\begin{vmatrix} 4 & 4 & 3600 \\ 1 & 1 & 750 \\ 4 & 0 & 2160 \end{vmatrix}}{D}$$

$$p = \frac{-5400}{-20} \quad c = \frac{-7200}{-20} \quad f = \frac{-2400}{-20}$$

$$p = 270 \quad c = 360 \quad f = 120$$

The diet includes 270 grams protein, 360 grams carbohydrates, and 120 grams fat.

Caution!

When an equation is missing one variable, be sure to write the missing term with a coefficient of zero.

$4p + 0c + 9f = 2160$

CHECK IT OUT! **4. What if...?** A diet requires 3200 calories, 700 grams of food, and 70% of the Calories from carbohydrates and fat. How many grams of protein, carbohydrates, and fat does the diet include?

THINK AND DISCUSS

1. Describe a matrix S that has no determinant.

2. Explain how you know what the three determinants will be when you apply Cramer's rule to a two-equation system in which one equation is a multiple of the other.

Know it! Note

3. **GET ORGANIZED** Copy and complete the graphic organizer. In each box, write the appropriate formula.

	2 × 2 Matrix	3 × 3 Matrix
Determinant		
Cramer's Rule		

4-4 Exercises

California Standards
2.0; Review of **7AF1.1**

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KEYWORD: MB7 Parent

GUIDED PRACTICE

1. **Vocabulary** Explain the meaning of a 0 entry in a *coefficient matrix*.

SEE EXAMPLE 1
p. 270

Find the determinant of each matrix.

2. $\begin{bmatrix} 7 & 5 \\ 9 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 1.5 & 0.25 \\ 6 & 2.5 \end{bmatrix}$ **4.** $\begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{3}{4} & -4 \end{bmatrix}$ **5.** $\begin{bmatrix} -3 & 40 \\ -5 & 66\frac{2}{3} \end{bmatrix}$

SEE EXAMPLE 2
p. 271

Use Cramer's rule to solve each system of equations.

6. $\begin{cases} 6x = 2 - y \\ 3x + 1 = 2y \end{cases}$ **7.** $\begin{cases} 4x + y + 6 = 0 \\ 8x + 2y = 9 \end{cases}$ **8.** $\begin{cases} 5x - 2y = 3 \\ 2.5x - y = 1.5 \end{cases}$ **9.** $\begin{cases} 2y = 2 - x \\ -3x + 6y = -9 \end{cases}$

SEE EXAMPLE 3
p. 272

Find the determinant of each matrix.

10. $P = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix}$ **11.** $S = \begin{bmatrix} 0 & -5 & -1 \\ 4 & 1 & 6 \\ 2 & 0.5 & 3 \end{bmatrix}$ **12.** $E = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

SEE EXAMPLE 4
p. 273

13. **Consumer** Naomi buys 2 pounds of trail mix, 1.5 pounds of mixed nuts, and 3 pounds of dried fruit for a total of $28.42. Briana buys 4.5 pounds of mixed nuts and 2 pounds of dried fruit for a total of $39.39. The price per pound of trail mix plus the price per pound of dried fruit is the same as the price per pound of mixed nuts. What is the price per pound of each product?

PRACTICE AND PROBLEM SOLVING

Find the determinant of each matrix.

14. $\begin{bmatrix} 3 & -0.4 \\ 5 & 0.3 \end{bmatrix}$ **15.** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **16.** $\begin{bmatrix} -\frac{2}{5} & 8 \\ -\frac{1}{2} & 10 \end{bmatrix}$ **17.** $\begin{bmatrix} r & -1 \\ -2r^2 & \pi r \end{bmatrix}$

Independent Practice	
For Exercises	See Example
14–17	1
18–21	2
22–24	3
25	4

Extra Practice

Skills Practice p. S11
Application Practice p. S35

Use Cramer's rule to solve each system of equations.

18. $\begin{cases} 0.5x + 6y = 2 \\ 0.25x + 3y = 0.5 \end{cases}$ **19.** $\begin{cases} x + 2y = 3.5 \\ 3x - y = 2.7 \end{cases}$ **20.** $\begin{cases} 2x + y = 3 \\ x + \dfrac{y}{2} = 2 \end{cases}$ **21.** $\begin{cases} 3y - x = 7 \\ 2x + 3y = -7 \end{cases}$

Find the determinant of each matrix.

22. $A = \begin{bmatrix} 2.5 & 1.5 & 0 \\ 3.2 & 1 & -4 \\ 6.4 & -5 & 2.1 \end{bmatrix}$ **23.** $L = \begin{bmatrix} -2.4 & 1 & 0 \\ 3 & 0 & 0.5 \\ 0 & 3.5 & 1 \end{bmatrix}$ **24.** $W = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -5 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

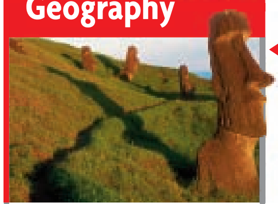
25. **Fitness** Cameron records the hours he exercises and the total Calories he burns each day. How many Calories are burned per hour for each of the three activities? Use Cramer's rule to solve.

Cameron's Activity Log

	Bicycling	Racquetball	Swimming	Calories Burned
Monday	1.5 h	1 h	0.75 h	1620
Wednesday	0.75 h	■	1 h	915
Friday	1 h	1.5 h	■	1230

LINK

Geography

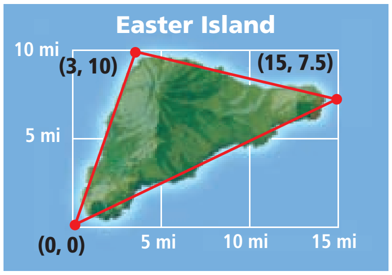


Easter Island, a South Pacific island of Chile, contains more than 600 stone statues. The statues were carved between 1600 and 1730. Most of the heads actually have torsos that have become buried over time.

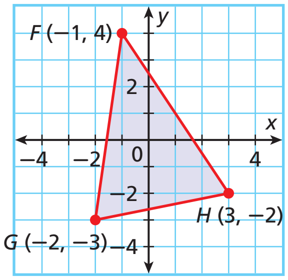
Geometry **The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is equal to the absolute value of A. Use this information for Exercises 26 and 27.**

$$A = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

26. **Geography** Find the area of Easter Island.



27. Find the area of $\triangle FGH$.



28. **Critical Thinking** For the system of equations $2x + y = 6$ and $cy = 3 - x$, for what value of c is the determinant zero? Explain your reasoning.

29. **Internet** John's site asks readers to rate his articles with 1, 2, or 3 points. There were 38 votes, twice as many 3's as 1's, and the point total was 85. How many people gave each rating?

Find the determinant of each matrix.

30. $A = \begin{bmatrix} x & x - 1 \\ x + 1 & x \end{bmatrix}$ **31.** $B = \begin{bmatrix} x - 2 & x + 2 \\ x + 2 & x + 6 \end{bmatrix}$ **32.** $C = \begin{bmatrix} 6x^2 & -6x + 2x^2 \\ 3x & x - 3 \end{bmatrix}$

33. **Currency** The United States Code specifies that dimes weigh 2.268 grams each and nickels weigh 5 grams each. The approximate weight of 425 dimes and nickels is 1483 grams.

a. How many of each coin are there?

b. What is the total value of the coins?

CONCEPT CONNECTION

34. This problem will prepare you for the Concept Connection on page 294.

At an amusement park, 6 Wild rides and 3 Mild rides require 48 tickets, while 2 Wild rides and 10 Mild rides require 52 tickets. Let x be the number of tickets for a Wild ride and y be the number of tickets for a Mild ride.

a. Write the problem as a system of equations.

b. Write the coefficient matrix, and find its determinant.

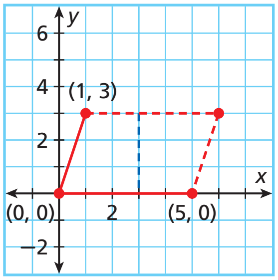
c. How many solutions are there?

d. Use Cramer's rule to find x and y. ⟶ $x = \dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D}$ and $y = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D}$

e. How many tickets are required for each ride?

35. Write About It Compare the process of deciding whether a proportion is true to the process of determining whether $D = 0$ for a 2×2 matrix.

36. Multi-Step The points $(5, 0)$ and $(1, 3)$ determine a parallelogram with respect to the origin as shown.



a. Find the area of the parallelogram.

b. Enter the two points in order into $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$, and evaluate. How does this value relate to the area of the parallelogram?

c. Change the width and height of the parallelogram, and find the area and the determinant. Does the relationship between the area and the determinant still hold?

d. Reverse the points in part **b** so that (x_1, y_1) is $(1, 3)$. Do the same for the parallelogram in part **c.** How does the order affect the determinant?

STANDARDIZED TEST PREP

37. Which of the following statements describes the system of equations $\begin{cases} 3x = y - 1 \\ x + 2y = 16 \end{cases}$?

Ⓐ Dependent; many solutions

Ⓑ Inconsistent; no solution

Ⓒ Inconsistent; many solutions

Ⓓ Consistent; one solution

38. Which matrix has a determinant of 1?

Ⓕ $\begin{bmatrix} 3 & 11 \\ 1 & 4 \end{bmatrix}$ Ⓖ $\begin{bmatrix} 3 & -11 \\ 1 & 4 \end{bmatrix}$ Ⓗ $\begin{bmatrix} -3 & 11 \\ 1 & 4 \end{bmatrix}$ Ⓙ $\begin{bmatrix} 3 & 11 \\ -1 & 4 \end{bmatrix}$

39. Gridded Response The determinant of $\begin{bmatrix} 4 & -5 \\ 1 & 2x \end{bmatrix}$ is 25. Find x.

CHALLENGE AND EXTEND

40. Suppose a 3×3 matrix has a row or column of zeros. Explain the effect on the determinant.

41. Write $x^2 + y^2$ as a determinant.

42. If $x = \dfrac{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}}{5}$ and $y = \dfrac{\begin{vmatrix} 7 & a \\ b & c \end{vmatrix}}{5}$, find the values of a, b, and c.

43. **Civics** A ballot measure received the vote percentages shown in the table. There were a total of 4826 votes. How many of the votes came from Southside?

Ballot Measure Voting		
District	**In Favor**	**Opposed**
Northside	47%	53%
Southside	85%	15%
Total	49%	51%

SPIRAL REVIEW

44. **Consumer Economics** Trish has $125 and a coupon for $10 off her total at the Toasty Coats Outlet. She finds a coat that is marked 25% off. Write an inequality for the maximum amount that the coat can be priced before the markdown so Trish can afford to buy it. *(Lesson 2-1)*

Use substitution to solve each system of equations. *(Lesson 3-2)*

45. $\begin{cases} x = \frac{1}{3}y \\ 6x - 6y = 16 \end{cases}$ **46.** $\begin{cases} x + y = -5 \\ 2x - y = -7 \end{cases}$ **47.** $\begin{cases} 2x = y \\ 4x + y = -2 \end{cases}$

Use a matrix to transform the polygon with coordinates $D(1, 1)$, $E(4, -2)$, $F(-2, -3)$, and $G(-1, -1)$. *(Lesson 4-3)*

48. Translate 5 units right and 3 units up.

49. Reflect *DEFG* across the x-axis.

50. Translate *DEFG* 1 unit left and 2 units down.

51. Dilate *DEFG* by a factor of 3.

Career Path

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KEYWORD: MB7 Career

Karen Michaels
Economist

Q: What math classes did you take in high school?

A: I took Algebra 1 and 2, Geometry, and Precalculus.

Q: What math classes did you take in college?

A: Math and economics are closely related, so I took several math classes—Statistics, Calculus, Mathematical Economics.

Q: Any topics you found particularly interesting?

A: Game theory. You wouldn't think it applies, but it has a lot of applications in math, economics, and political science. It's about how people make decisions that affect other people.

Q: How do you use math as an economist?

A: I've conducted research projects on energy costs, interest rates, inflation, and employment levels. I collect, analyze, and summarize data and forecast economic trends.

4-5 Matrix Inverses and Solving Systems

Objectives
Determine whether a matrix has an inverse.

Solve systems of equations using inverse matrices.

Vocabulary
multiplicative inverse matrix
matrix equation
variable matrix
constant matrix

California Standards

2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or **with matrices.**

Who uses this?
Cryptographers, who create and crack codes, may use matrices to protect the privacy of messages. (See Example 4.)

You can encode a message using a matrix. The receiver can use an inverse process to decode your message.

A matrix can have an inverse only if it is a square matrix. But not all square matrices have inverses. If the product of the square matrix A and the square matrix A^{-1} is the identity matrix I, then $AA^{-1} = A^{-1}A = I$, and A^{-1} is the **multiplicative inverse matrix** of A, or just the *inverse* of A.

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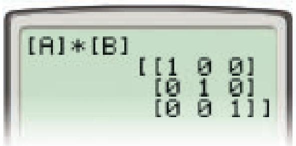
EXAMPLE 1 Determining Whether Two Matrices Are Inverses

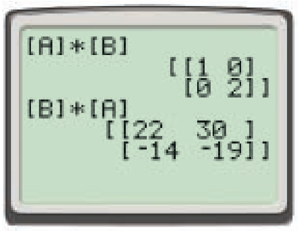
Determine whether the two given matrices are inverses.

A $\begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 2 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{6} \\ -2 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$



The product is the identity matrix I, so the matrices are inverses.

B $\begin{bmatrix} 2 & 3 \\ 7 & 10 \end{bmatrix}$ and $\begin{bmatrix} -10 & 6 \\ 7 & -4 \end{bmatrix}$



Neither product is I, so the matrices are not inverses.

Remember!
The identity matrix I has 1's on the main diagonal and 0's everywhere else.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

CHECK IT OUT!

1. Determine whether the given matrices are inverses.

$$\begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -0.2 & 0 & 0.4 \\ 1.2 & 1 & -1.4 \\ 0.4 & 0 & 0.2 \end{bmatrix}$$

Inverse of a 2 × 2 Matrix

The inverse of a 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If the determinant is 0, $\dfrac{1}{\det A}$ is undefined. So a matrix with a determinant of 0 has no inverse. It is called a *singular* matrix.

Helpful Hint

To find $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ from $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, think "switch ops" for the cross products. *Switch* a and d. Take the *opposites* of b and c.

EXAMPLE 2 Finding the Inverse of a 2 × 2 Matrix

Find the inverse of the matrix, if it is defined.

A $A = \begin{bmatrix} -2 & 2 \\ 3 & -4 \end{bmatrix}$

First, check that the determinant is nonzero. The determinant is $(-2)(-4) - 3(2) = 8 - 6 = 2$, so the matrix has an inverse.

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is $\dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

So the inverse of $A = \begin{bmatrix} -2 & 2 \\ 3 & -4 \end{bmatrix}$ is $A^{-1} = \dfrac{1}{2}\begin{bmatrix} -4 & -2 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -\frac{3}{2} & -1 \end{bmatrix}$.

Use a calculator to check, as in Example 1.

B $B = \begin{bmatrix} \frac{1}{2} & 2 \\ 3 & 12 \end{bmatrix}$

The determinant is $\frac{1}{2}(12) - 3(2) = 0$, so B has no inverse.

CHECK IT OUT! **2.** Find the inverse of $\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$, if it is defined.

You can use the inverse of a matrix to solve a system of equations. This process is similar to solving an equation such as $5x = 20$ by multiplying each side by $\frac{1}{5}$, the multiplicative inverse of 5.

To solve systems of equations with the inverse, you first write the **matrix equation** $AX = B$, where A is the coefficient matrix, X is the **variable matrix**, and B is the **constant matrix**.

The matrix equation representing $\begin{cases} x + y = 8 \\ 2x + y = 1 \end{cases}$ is shown.

$$A \cdot X = B$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Coefficient matrix A Variable matrix X Constant matrix B

To solve $AX = B$, multiply both sides by the inverse A^{-1}.

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B \quad \textit{The product of } A^{-1} \textit{ and } A \textit{ is } I.$$

$$X = A^{-1}B$$

EXAMPLE 3 Solving Systems Using Inverse Matrices

Write the matrix equation for the system, and solve.

$$\begin{cases} x + y = 8 \\ 2x + y = 1 \end{cases}$$

Step 1 Set up the matrix equation.

$$\begin{matrix} A & X & = & B \end{matrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Write: coefficient matrix • variable matrix = constant matrix.

Step 2 Find the determinant.

The determinant of A is $1 - 2 = -1$.

Step 3 Find A^{-1}.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \text{ so } A^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\begin{matrix} X & = & A^{-1} & B \end{matrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \end{bmatrix} \quad \textit{Multiply.}$$

$$= \begin{bmatrix} -7 \\ 15 \end{bmatrix}$$

The solution is $(-7, 15)$.

> **Caution!**
> Matrix multiplication is not commutative, so it is important to multiply by the inverse *in the same order* on both sides of the equation. A^{-1} comes *first* on each side.

CHECK IT OUT! **3.** Write the matrix equation for $\begin{cases} x + y = 4 \\ 2x + 3y = 9 \end{cases}$ and solve.

EXAMPLE 4 *Problem-Solving Application: Cryptography*

PROBLEM SOLVING

You receive a coded instant message from Lupe.

Both you and Lupe use the same encoding matrix $E = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$.

Upon decoding the message, you will get a matrix where letters are represented by numbers (*A* is 1, *B* is 2, ... *Z* is 26, and 0 is a space). Decode the message.

1 Understand the Problem

The **answer** will be the words of the message, uncoded.

List the important information:

- The encoding matrix is E.
- Lupe used M as the message matrix, with letters written as the integers 0 to 26, and then used EM to create the two-row code matrix C.

$$C = \begin{bmatrix} 240 & 48 & 70 & 5 & 173 & 6 & 245 & 183 & 159 \\ 284 & 56 & 83 & 6 & 205 & 7 & 290 & 216 & 189 \end{bmatrix}$$

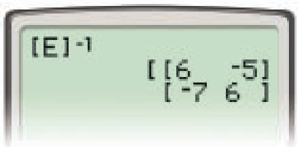
2 Make a Plan

Because $EM = C$, you can use $M = E^{-1}C$ to decode the message into numbers and then convert the numbers to letters.

- Multiply E^{-1} by C to get M, the message written as numbers.
- Use the letter equivalents for the numbers in order to write the message as words so that you can read it.

3 Solve

Use a calculator to find E^{-1}.



$$E^{-1} = \begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix}$$

Multiply E^{-1} by C.

20 = T, and so on

$$M = E^{-1}C = \begin{bmatrix} 20 & 8 & 5 & 0 & 13 & 1 & 20 & 18 & 9 \\ 24 & 0 & 8 & 1 & 19 & 0 & 25 & 15 & 21 \end{bmatrix}$$

T H E _ M A T R I
X _ H A S _ Y O U

The message in words is "The matrix has you."

4 Look Back

You can verify by multiplying E by M to see that the decoding was correct. If the math had been done incorrectly, getting a different message that made sense would have been very unlikely.

Helpful Hint

To reduce rounding errors, enter fractions when appropriate. The calculator will convert your entries to decimal representations.

CHECK IT OUT! **4.** Use the encoding matrix $E = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ to decode this message.

THINK AND DISCUSS

1. Explain what the existence of the inverse of matrix S, S^{-1}, tells you about matrix S.

2. Describe the inverse of an identity matrix.

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer. Compare multiplicative inverses of real numbers and matrices.

Multiplicative Inverses		
	Real Numbers	**Matrices**
Notation and Example		
How to Show That It Is the Multiplicative Inverse		
Commutative Property		

4-5 Exercises

California Standards
2.0; Review of 1A5.0

go.hrw.com
Homework Help Online
KEYWORD: MB7 4-5
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

1. **Vocabulary** Describe how to create a *matrix equation* from a system of equations.

SEE EXAMPLE 1 p. 278

Determine whether the given matrices are inverses.

2. $\begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{8} & \frac{3}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0.4 & 1 \\ 1.2 & 0 & 0.8 \\ -1.6 & 0.2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 12.5 & 2 \\ -1.6 & 2 & -1 \\ 5 & 1 & -10 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

SEE EXAMPLE 2 p. 279

Find the inverse of the matrix, if it is defined.

5. $\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$

6. $\begin{bmatrix} 1 & 7 \\ 2 & 6 \end{bmatrix}$

7. $\begin{bmatrix} \frac{1}{3} & 2 \\ \frac{3}{2} & 9 \end{bmatrix}$

8. $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 8 & 7 \\ 9 & 8 \end{bmatrix}$

SEE EXAMPLE 3 p. 280

Write the matrix equation for the system, and solve.

10. $\begin{cases} 3x - y = 5 \\ y = 2x - 4 \end{cases}$

11. $\begin{cases} 5x + 9y = 1 \\ 2 - 4x - 7y = 4 \end{cases}$

12. $\begin{cases} 2x + 4y = 3 \\ 2x + 3y = 1 \end{cases}$

SEE EXAMPLE 4 p. 280

13. **Cryptography** Rayanne receives the message shown, giving Sara's current location somewhere in Asia. The message was encoded using $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$. Write the decoding matrix, and decode the message.

PRACTICE AND PROBLEM SOLVING

Independent Practice	
For Exercises	See Example
14–16	1
17–21	2
22–24	3
25	4

Extra Practice
Skills Practice p. S11
Application Practice p. S35

Determine whether the given matrices are inverses.

14. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

15. $\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{4} & -2 \end{bmatrix}\begin{bmatrix} -\frac{16}{15} & -\frac{4}{15} \\ -\frac{2}{15} & -\frac{8}{15} \end{bmatrix}$

16. $\begin{bmatrix} 1 & 5 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0.2 & -0.2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Find the inverse of the matrix, if it is defined.

17. $\begin{bmatrix} -0.25 & -0.5 \\ -1.5 & -2 \end{bmatrix}$

18. $\begin{bmatrix} 7 & 14 \\ 3 & 6 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$

20. $\begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$

21. $\begin{bmatrix} -2 & -3 \\ 7 & 11 \end{bmatrix}$

Write the matrix equation for the system, and solve.

22. $\begin{cases} x - y = 5 \\ 2y - x = 6 \end{cases}$

23. $\begin{cases} x + 2y = 6 \\ 2x + y = 9 \end{cases}$

24. $\begin{cases} 4x + 7y = 10 \\ 3x + 5y = 9 \end{cases}$

25. **Cryptography** Quinn receives the coded message shown, which tells him when he needs to report to headquarters. It was encoded using the matrix $\begin{bmatrix} 7 & 3 \\ 9 & 4 \end{bmatrix}$. Write the decoding matrix, and decode the message. When will Quinn need to report?

26. **Packaging** Cara compares three fruit and nut gift packs. Write the matrix equation and solve to find the cost per pound of pears, pecans, and nectarines.

27. **Multi-Step** On an outdoor trip, the organizers take seven inflatable boats, 6-person boats and 2-person boats, for 34 people. The system of equations that represents this situation is $\begin{cases} 6x + 2y = 34 \\ x + y = 7 \end{cases}$, where x represents the number of 6-person boats and y the number of 2-person boats.

   a. Write the coefficient matrix.

   b. Write the appropriate matrix equation.

   c. Find the inverse of the coefficient matrix.

   d. Solve the matrix equation to find how many of each size boat the group takes.

28. **Critical Thinking** How are the inverse matrix and identity matrix related?

29. E is an encoding matrix for message M that gives a coded message C. What are the dimension restrictions on E, M, and C?

30. **ERROR ANALYSIS** Which inverse is incorrect for $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$? Explain the error.

A $\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$

B $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}$

31. **Entertainment** A game show host says that he has $5000 in $50 bills and $100 bills and he will give you the $5000 if you can tell him how many of each type of bill he has. He gives you a hint that he has 73 bills. Use an inverse matrix to find how many of each he has.

32. **Water** A fountain operating 24 hours a day can be set at three different speeds, low, medium, and high. Find the number of kL/h the fountain uses at each speed.

	Time on Low (h)	Time on Med (m)	Time on High (h)	Kiloliters Used
Monday	15	7	2	199
Tuesday	16	4	4	208
Wednesday	12	8	4	236

33. **What if...?** Suppose the entries of $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ are doubled.

   a. What happens to the entries of the inverse matrix?

   b. Suppose the entries of a square matrix are multiplied by n. Make a conjecture about the entries of the inverse matrix.

CONCEPT CONNECTION

34. This problem will prepare you for the Concept Connection on page 294.

At a carnival, 2 meals and 7 rides require 24 tickets, while 4 meals and 13 rides require 46 tickets. Let x be the number of tickets for a meal and y be the number of tickets for a ride.

a. Write the problem as a system of equations.

b. Is the determinant $D = 0$? How many solutions are there?

c. Write the coefficient matrix, and find its inverse.

d. Use $X = A^{-1}B$ to find x and y.

e. How many tickets are required for each item?

35. **a.** **Critical Thinking** Prove that the inverse of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

b. If the determinant of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 1, what is its inverse?

c. If a, b, c, and d are integers, why does the inverse contain only integers?

36. Complete the matrix $\begin{bmatrix} 2 & ? \\ 4 & 3 \end{bmatrix}$ so that it has no inverse.

37. Suppose A is the 1-entry matrix $\begin{bmatrix} a \end{bmatrix}$. What is its inverse?

38. **Chemistry** A laboratory has one solution of 15% hydrochloric acid (HCl) and one solution of 40% HCL. A mixture requires 50 liters of 35% HCL. How many liters of each must be used?

39. **Write About It** Find the product of

$\begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$ and $\begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix}$.

Describe the relationship between these matrices.

STANDARDIZED TEST PREP

40. Which is the correct matrix equation for the system $\begin{cases} 3x + 2y = 8 \\ x = y + 1 \end{cases}$?

(A) $\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(B) $\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(D) $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

41. Which statement is a true statement about matrix $G = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}$?

(F) G has an inverse because the determinant is NOT 0.

(G) G has an inverse because the determinant is 0.

(H) G has no inverse because the determinant is 0.

(J) G has no inverse because the determinant is NOT 0.

42. B is the inverse of $\begin{bmatrix} -1 & 6 \\ 4 & 3 \end{bmatrix}$. What is entry b_{11}?

(A) 1 (B) $-\frac{1}{9}$ (C) 3 (D) $-\frac{1}{27}$

43. In matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a > 0$, $b < 0$, $c < 0$, $d > 0$, and det $A \neq 0$. Which of the following is true?

Ⓕ A^{-1} has no negative entries.

Ⓖ A^{-1} has one negative entry.

Ⓗ A^{-1} has two negative entries.

Ⓙ A^{-1} has three negative entries.

44. **Extended Response** An art gallery gives away small prints valued at \$25 for donations of \$500, and larger prints valued at \$50 for donations of \$1000 and above. The gallery raises \$24,000 and gives away 35 prints. Find the number of each size print that the gallery gives away.

CHALLENGE AND EXTEND

45. **Hobbies** A fantasy league rating system rates NBA point guards by assigning a rating multiplier to each of the following categories: points per game, assists per game, turnovers per game, and steals per game. What multiplier is assigned to each category?

Point Guard Ratings 2004–2005

Point Guard	Points/Game	Assists/Game	Turnovers/Game	Steals/Game	Rating
Nash	15.5	11.5	3.3	1.0	78.3
Marbury	21.7	8.2	2.8	1.5	77.7
B. Davis	19.2	7.9	2.9	1.8	71.7
Kidd	14.4	8.3	2.5	1.9	66.0

46. For what values of e, f, g, and h will matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ be its own inverse?

47. Quinn uses a 3×3 decoding matrix on the message shown, where each entry on the main diagonal and above it is 1 and each entry below the main diagonal is 0.

a. What message did he receive?

b. What encoding matrix does he use?

c. He sends the reply "I will try" by using the corresponding encoding matrix. What coded message does he send?

SPIRAL REVIEW

Solve. *(Lesson 2-2)*

48. $\frac{12}{30} = \frac{2x}{10}$

49. $\frac{100}{7} = \frac{0.5}{0.2x}$

50. 125% of $x = 117$

Use elimination to solve each system of equations. *(Lesson 3-6)*

51. $\begin{cases} x + y - z = 2 \\ 2x + 3y - 6z = 5 \\ -4x - 5y + 0.25z = -9 \end{cases}$

52. $\begin{cases} y - x - 3z = 4 \\ 2x + y - 4z = -3 \\ 0.25x + 8z + 3 = 2y \end{cases}$

Find the determinant of each matrix. *(Lesson 4-4)*

53. $\begin{bmatrix} 5 & -6 \\ 1 & 0.5 \end{bmatrix}$

54. $\begin{bmatrix} \frac{1}{6} & 3 \\ 1 & 12 \end{bmatrix}$

55. $\begin{bmatrix} -4 & 1 & 6 \\ 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$

56. $\begin{bmatrix} \frac{4}{9} & 8 \\ \frac{3}{2} & -81 \end{bmatrix}$

4-5 Technology LAB

Use Spreadsheets with Matrices to Solve Systems

You can use matrix inversion on a spreadsheet to solve systems of equations.

Use with Lesson 4-5

California Standards

2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or **with matrices.**

Activity

Solve the system $\begin{cases} 7x + 2y = -8 \\ -3x + y = 9 \end{cases}$.

You can find determinants and inverses and solve $X = A^{-1}B$ by using a spreadsheet. To find A^{-1}, first find the determinant of A by using the spreadsheet (subtracting cross products).

Enter the four coefficients of the constant matrix into cells B2, C2, B3, and C3. Calculate its determinant by entering **=B2*C3–B3*C2** in cell C5.

C5 fx =B2*C3-B3*C2

	A	B	C	D
1				
2	Matrix A	7	2	
3		-3	1	
4				
5	Determinant A		13	

The inverse of a 2 × 2 matrix is $\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, or $\begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$.

Begin with cell C7, and enter the formula for the first entry **=C3/C5**. Enter the formulas for the other three entries, D7: **=–C2/C5**, C8: **=–B3/C5**, D8: **=B2/C5**.

The solution is the 2 × 1 matrix $A^{-1}B$. Enter the constant matrix in cells E7 and E8. Multiply A^{-1} by B by entering **=C7*E7+D7*E8** in cell D10 and **=C8*E7+D8*E8** in cell D11.

fx =C7*E7+D7*E8

	A	B	C	D	E
6					Matrix B
7	Inverse A or A^{-1}		0.076923	-0.15385	-8
8			0.230769	0.538462	9
9					
10		Solution $A^{-1}B$		-2	
11					

The solution is $x = -2$ and $y = 3$. You now have a solving "machine" for any 2 × 2 system. See what happens when you change one or more entries in A or in the constant matrix.

D11 fx =C8*E7+D8*E8

	A	B	C	D	E
6					Matrix B
7	Inverse A or A^{-1}		0.076923	-0.15385	-8
8			0.230769	0.538462	9
9					
10		Solution $A^{-1}B$		-2	
11				3	

Try This

1. Change the constants to –5 and 9, and solve the system by using a spreadsheet.
2. How can you check your answers by using the spreadsheet?
3. **Critical Thinking** Solve a system you know to be inconsistent by using the spreadsheet. Solve a system you know to be dependent. How can you tell from the spreadsheet whether a system is inconsistent or dependent?

4-6 Row Operations and Augmented Matrices

Objective
Use elementary row operations to solve systems of equations.

Vocabulary
augmented matrix
row operation
row reduction
reduced row-echelon form

California Standards
2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or **with matrices.**

Who uses this?
Workers at an animal shelter can use augmented matrices to analyze the contents of shipments. (See Example 3.)

In previous lessons, you saw how Cramer's rule and inverses can be used to solve systems of equations. Solving large systems requires a different method using an *augmented matrix*. An **augmented matrix** consists of the coefficients and constant terms of a system of linear equations.

$$\begin{cases} 7x + 3y = 4 \\ 2x - 3y = 10 \end{cases} \qquad \left[\begin{array}{cc|c} 7 & 3 & 4 \\ 2 & -3 & 10 \end{array}\right]$$

A vertical line separates the coefficients from the constants.

EXAMPLE 1 Representing Systems as Matrices

Write the augmented matrix for the system of equations.

A $\begin{cases} -3y = x + 12 \\ -2y = 7 \end{cases}$

Step 1 Write each equation in $ax + by = c$ form.

$$-x - 3y = 12$$
$$0x - 2y = 7$$

Step 2 Write the augmented matrix, with coefficients and constants.

$$\left[\begin{array}{cc|c} -1 & -3 & 12 \\ 0 & -2 & 7 \end{array}\right]$$

B $\begin{cases} x - y = 5 \\ z - x = 7 \\ y = z + 6 \end{cases}$

Step 1 Write each equation in $Ax + By + Cz = D$ form.

$$x - y + 0z = 5$$
$$-x + 0y + z = 7$$
$$0x + y - z = 6$$

Step 2 Write the augmented matrix, with coefficients and constants.

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ -1 & 0 & 1 & 7 \\ 0 & 1 & -1 & 6 \end{array}\right]$$

CHECK IT OUT! **Write the augmented matrix.**

1a. $\begin{cases} -x = y \\ 2 - y = x \end{cases}$

1b. $\begin{cases} -5x - 12 = 4y \\ z = 3 - x \\ 10 = 3z + 4y \end{cases}$

You can use the augmented matrix of a system to solve the system. First you will do a **row operation** to change the form of the matrix. These row operations create a matrix equivalent to the original matrix. So the new matrix represents a system equivalent to the original system.

For each matrix, the following row operations produce a matrix of an equivalent system.

Know it! Note

Elementary Row Operations	
• Switch any two rows.	$\left[\begin{array}{cc:c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc:c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right]$
• Multiply a row by a nonzero constant.	$\left[\begin{array}{cc:c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc:c} 2 & 4 & 6 \\ 4 & 5 & 6 \end{array}\right]$
• Replace a row with the sum or difference of that row and another row. $\left[\begin{array}{cc:c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc:c} 1 & 2 & 3 \\ 1+4 & 2+5 & 3+6 \end{array}\right]$	
• Combine these operations.	

Row reduction is the process of performing elementary row operations on an augmented matrix to solve a system. The goal is to get the coefficients to reduce to the identity matrix on the left side.

This is called **reduced row-echelon form**. $\left[\begin{array}{cc:c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right] \begin{array}{l} \rightarrow 1x = 5 \\ \rightarrow 1y = 2 \end{array}$

EXAMPLE 2 Solving Systems with an Augmented Matrix

Write the augmented matrix, and solve.

A $\begin{cases} 6x + y = 9 \\ 3x + 2y = 0 \end{cases}$

Step 1 Write the augmented matrix. $\left[\begin{array}{cc:c} 6 & 1 & 9 \\ 3 & 2 & 0 \end{array}\right]$

Step 2 Multiply row 2 by 2.

$$\left[\begin{array}{cc:c} 6 & 1 & 9 \\ 3 & 2 & 0 \end{array}\right] \quad 2 ② \rightarrow \left[\begin{array}{cc:c} 6 & 1 & 9 \\ 6 & 4 & 0 \end{array}\right]$$

Step 3 Subtract row 1 from row 2. Write the result in row 2.

$$② - ① \rightarrow \left[\begin{array}{cc:c} 6 & 1 & 9 \\ 0 & 3 & -9 \end{array}\right]$$

Although row 2 is now $3y = -9$, an equation easily solved for y, row operations can be used to solve for both variables.

Step 4 Multiply row 1 by 3.

$$3 ① \rightarrow \left[\begin{array}{cc:c} 18 & 3 & 27 \\ 0 & 3 & -9 \end{array}\right]$$

Step 5 Subtract row 2 from row 1. Write the result in row 1.

$$① - ② \rightarrow \left[\begin{array}{cc:c} 18 & 0 & 36 \\ 0 & 3 & -9 \end{array}\right]$$

Remember!

2 ② is read as "2 times row 2." ② − ① is read as "row 2 minus row 1."

Step 6 Divide row 1 by 18 and row 2 by 3.

$$\begin{array}{l} ❶ \div 18 \rightarrow \\ ❷ \div 3 \rightarrow \end{array} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \begin{array}{l} \rightarrow 1x = 2 \\ \rightarrow 1y = -3 \end{array}$$

The solution is $x = 2$, $y = -3$. Check the result in the original equations.

Write the augmented matrix and solve.

B $\begin{cases} x + y = 5 \\ 3x + 3y = 7 \end{cases}$

$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 3 & 3 & 7 \end{array}\right]$ *Write the augmented matrix.*

$3❶ \rightarrow \left[\begin{array}{cc|c} 3 & 3 & 15 \\ 3 & 3 & 7 \end{array}\right]$ $\quad$ $❷ - ❶ \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 0 & -8 \end{array}\right]$

The second row means $0 + 0 = -8$, which is always false.
The system is inconsistent.

C $\begin{cases} -4y = 1 - 6x \\ 3x = 2y + \frac{1}{2} \end{cases}$

Write each equation in standard form.

$\begin{cases} 6x - 4y = 1 \\ 3x - 2y = \frac{1}{2} \end{cases}$

$\left[\begin{array}{cc|c} 6 & -4 & 1 \\ 3 & -2 & \frac{1}{2} \end{array}\right]$ *Write the augmented matrix.*

$2❷ - ❶ \rightarrow \left[\begin{array}{cc|c} 6 & -4 & 1 \\ 0 & 0 & 0 \end{array}\right]$

The second row means $0 + 0 = 0$, which is always true.
The system is dependent.

CHECK IT OUT! **Write the augmented matrix, and solve.**

2a. $\begin{cases} 4x + 4y = 32 \\ x + 3y = 16 \end{cases}$ $\quad$ **2b.** $\begin{cases} 3y = 15 - 9x \\ -6x = 2y + 10 \end{cases}$

On many calculators, you can add a column to a matrix to create the augmented matrix and can use the row reduction feature. So, the matrices in the Check It Out problem are entered as 2×3 matrices.

Student to Student

Solving Systems of Equations

Marcus Barrett
Memorial High School

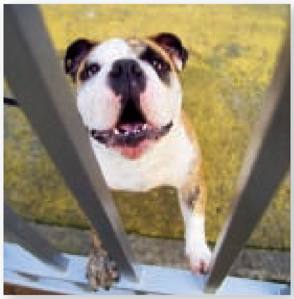
I'm glad I learned all of the different methods for solving systems, but if I have a graphing calculator available, I prefer $A^{-1}B$. At first I thought, "Why'd they wait so long to give us this?"

Without a graphing calculator or a spreadsheet, I'd use elimination for most cases.

Another thing I might do—I might use a spreadsheet on my computer, find determinants, and use Cramer's rule. Cramer's rule is good when you just want the value of one variable.

Now, if I had to solve a 20 by 20 system...

EXAMPLE 3 Charity Application



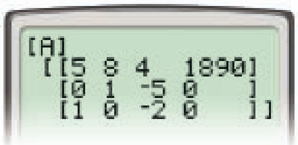
An animal shelter receives a shipment of items worth a total of $1890. Large bags of dog food are $8 each, pet blankets are $5 each, and dog toys are $4 each. There are 5 bags of dog food for each dog toy and twice as many blankets as dog toys. How many of each item are in the shipment? Solve by using row reduction on a calculator.

Use the facts to write three equations.

$5b + 8d + 4t = 1890$ *b = blankets*

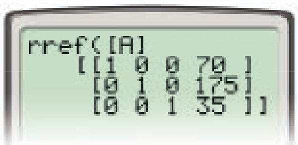
$d - 5t = 0$ *d = bags of dog food*

$b - 2t = 0$ *t = toys*



Enter the 3×4 augmented matrix as A.

Press 2nd x^{-1} (MATRX), select **MATH**, and move down the list to **B:rref(** to find the reduced row-echelon form of the augmented matrix.



There are 70 blankets, 175 bags of dog food, and 35 toys.

CHECK IT OUT!

3a. Solve by using row reduction on a calculator. $\begin{cases} 3x - y + 5z = -1 \\ x + 2z = 1 \\ x + 3y - z = 25 \end{cases}$

3b. A new freezer costs $500 plus $0.20 a day to operate. An old freezer costs $20 plus $0.50 a day to operate. After how many days is the cost of operating each freezer equal? Solve by using row reduction on a calculator.

THINK AND DISCUSS

1. Explain what the rows $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 3 \end{array}\right]$ and $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 3 \end{array}\right]$ tell you about a system of equations when you solve a system of three equations by using augmented matrices and reduced row-echelon form.

2. Tell how you know when an augmented matrix is in reduced row-echelon form.

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer. Fill in the augmented matrix for a three-equation system. Then write an example of the given operation in each box. Tell whether the operation produces an equivalent system.

	System of Equations	Augmented Matrix
Interchange rows or equations.		
Replace a row or equation with a multiple.		
Replace a row or equation with a sum or difference.		
Combine the above.		

4-6 Exercises

California Standards
2.0 Preparation for 9.0

go.hrw.com
Homework Help Online
KEYWORD: MB7 4-6
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

1. **Vocabulary** In an *augmented matrix*, where do you place the coefficients of the variables from the related system of equations?

SEE EXAMPLE 1 p. 287

Write the augmented matrix for each system of equations.

2. $\begin{cases} y - 3 = 2x \\ 3x = -y \end{cases}$

3. $\begin{cases} x + y + z = 10 \\ 2x + z = 12 \\ z - y = 3 \end{cases}$

4. $\begin{cases} 2x - 9 = y \\ 2z = 3y + 7 \\ z = 6 - x \end{cases}$

5. $\begin{cases} y + 2 = 3x \\ \frac{1}{4}y = z - 1 \\ z - 8 = \frac{x}{2} \end{cases}$

SEE EXAMPLE 2 p. 288

Write the augmented matrix, and use row reduction to solve.

6. $\begin{cases} 2y = x + 1 \\ 3x - 2 = y \end{cases}$

7. $\begin{cases} 8y = x + 7 \\ 3y + \frac{x}{2} = 0 \end{cases}$

8. $\begin{cases} x = 2y + 3 \\ y = \frac{1}{2}(x - 3) \end{cases}$

9. $\begin{cases} y = 4 + x \\ 4y - 3 = 4x \end{cases}$

SEE EXAMPLE 3 p. 290

10. **School** During a game, high school students sell snacks. They sell cold sandwiches for \$2.50, hot dogs for \$1.50, and hamburgers for \$2. By the end of the day, the students have collected \$1060.50 and sold 562 items. Casey estimates that the students sold twice as many hot dogs as cold sandwiches. If his estimate is correct, how many of each item did they sell? Solve by using row reduction on a calculator.

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
11–13	1
14–16	2
17	3

Extra Practice
Skills Practice p. S11
Application Practice p. S35

Write the augmented matrix for each system of equations.

11. $\begin{cases} \frac{1}{2}(x + 3y) = z \\ y = 2x + 4 \\ x + y + z = 3 \end{cases}$

12. $\begin{cases} 2y + z = 5 \\ y = 2z \end{cases}$

13. $\begin{cases} 0.1x + 0.2y + 0.15z = 1.0 \\ x + y = z \\ 2y = 1.3x \end{cases}$

Write the augmented matrix, and use row reduction to solve.

14. $\begin{cases} y + 2z = 9 \\ 2y + 4z = 13 \end{cases}$

15. $\begin{cases} 5x = y + 2 \\ y - x = 4 \end{cases}$

16. $\begin{cases} x + y = 4 \\ 3x = 9 - 2y \end{cases}$

17. **Math History** The Hundred Fowl problem asks, "A rooster is worth 5 coins, a hen 3 coins, and 3 chicks 1 coin. With 100 coins, we buy 100 of them. How many roosters, hens, and chicks are there?" There are seven times as many chicks as roosters. Write a set of equations and an augmented matrix for this problem. Solve by using row reduction on a calculator.

18. **Geometry** Write an augmented matrix to find the point of intersection of the two lines given by the equations $5y + 4x = 25$ and $y = 3x - 14$. Solve by using row reduction.

Write a system of equations for each augmented matrix.

19. $\left[\begin{array}{cc|c} 2 & 5 & -4 \\ 0 & 1 & -2 \end{array}\right]$

20. $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -2 \\ -1 & 9 & 1 & -9 \end{array}\right]$

21. $\left[\begin{array}{ccc|c} 0 & -1 & 0 & 3 \\ -7 & 0 & 2 & 0 \\ 0 & 0 & -10 & 4 \end{array}\right]$

Write the augmented matrix, and use row reduction to solve.

22. $\begin{cases} 2x + 5y = 8 \\ y - x = 10 \end{cases}$

23. $\begin{cases} 3x - y = -9 \\ 7y - 4x = 12 \end{cases}$

24. $\begin{cases} 3y = x + 5 \\ 9y - 3x = 15 \end{cases}$

25. $\begin{cases} 2x + 5y = z \\ 3y + 7 = x \\ x + 7z = 25 \end{cases}$

LINK

Math History

The ancient Chinese were fascinated with mathematical puzzles, such as tangrams, which were used to form many shapes.

26. **Math History** Around the second century B.C.E., a Chinese mathematician posed a problem. He set up a table to show different combinations—A, B, C—of bundles of three types of corn—1, 2, 3—and found the number of measures of corn in each bundle. Use an augmented matrix to solve this problem.

Chinese Math Puzzle

	A	B	C
Type-1 Bundles	3	2	1
Type-2 Bundles	2	3	2
Type-3 Bundles	1	1	3
Total Measures of Corn	39	34	26

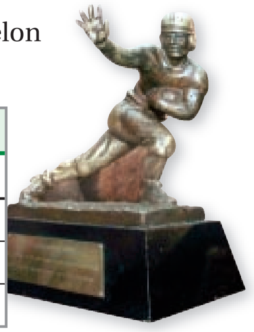
27. **Multi-Step** Voting data for the 2003 Heisman Trophy is given in the table.

a. Write a system of equations to represent the data.

b. Solve by using an augmented matrix. Show it in reduced row-echelon form. Find the number of points that each vote is worth.

2003 Heisman Trophy Votes

Player	First Place	Second Place	Third Place	Points
Jason White	319	204	116	1481
Larry Fitzgerald	253	233	128	1353
Eli Manning	95	132	161	710



28. **Write About It** Explain the difference between a coefficient matrix and an augmented matrix.

Solve the system by using row reduction on a calculator.

29. $\begin{cases} 3x = 5 - 4z \\ x + y + z = 5 \\ y = 2z \end{cases}$

30. $\begin{cases} x + y = z \\ 5y - 2z = 4 \\ 5y - 2x = 8 \end{cases}$

31. $\begin{cases} 2x + y - z = 5 \\ z = -2x - y \\ y = x \end{cases}$

32. **Critical Thinking** How can you identify a dependent or inconsistent system by looking at an augmented matrix in reduced row-echelon form?

CONCEPT CONNECTION

33. This problem will prepare you for the Concept Connection on page 294.

At a carnival, 3 meals and 8 rides require 64 tickets, while 4 meals and 11 rides require 87 tickets. Let x be the number of tickets for a meal and y be the number of tickets for a ride.

a. Write the problem as a system of equations.

b. Write the augmented matrix.

c. Use row reduction to solve.

d. How many tickets are required for each item?

34. **Photography** The yearbook photographer sells sets of photos in three sizes. The price of each set includes a base price and the price for each size of print. The base price is twice the price of a large print. Find the base price and the price for each size of print.

STANDARDIZED TEST PREP

35. Which operation cannot be used to solve a system of equations by using an augmented matrix and row reduction?

Ⓐ Multiply any two rows together.
Ⓑ Subtract one row from another.
Ⓒ Switch any two rows.
Ⓓ Multiply a row by a constant.

36. Which row-reduced matrix indicates a dependent system of equations?

Ⓕ $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$

Ⓖ $\left[\begin{array}{cc|c} 4 & 5 & 7 \\ 0 & 0 & \frac{2}{3} \end{array}\right]$

Ⓗ $\left[\begin{array}{cc|c} 4 & 5 & 7 \\ 0 & 0 & \frac{0}{5} \end{array}\right]$

Ⓙ $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

37. Which is the solution to the system represented by $\left[\begin{array}{cc|c} 2 & 0 & 5 \\ 0 & 3 & -3 \end{array}\right]$?

Ⓐ (5, −3)
Ⓑ (2.5, −3)
Ⓒ (2.5, −1)
Ⓓ (5, −1)

CHALLENGE AND EXTEND

38. Write an augmented matrix in which transposing two rows would be the best first step. Justify your reasoning.

39. The system represented by $\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 3 & 1 & 8 \\ -2 & 4 & -10 \end{array}\right]$ has a solution. Explain why.

SPIRAL REVIEW

Describe each transformation of $f(x) = x^3$. *(Lesson 1-9)*

40. $f(x) = x^3 - 5$

41. $f(x) = \frac{3}{8}x^3$

42. $f(x) = (x + 3)^3$

43. Maximize $P = 3x + 2y$ given the constraints $x \geq 0$, $y \geq 0$, $x \leq y$ and $-2x + 3 \geq y$, and identify the point where P is maximized. *(Lesson 3-4)*

Write the matrix equation for the system, and solve. *(Lesson 4-5)*

44. $\begin{cases} 5y = x + 12 \\ 2y = 2x + 8 \end{cases}$

45. $\begin{cases} 3x - y = 0 \\ x + 2y = 7 \end{cases}$

CHAPTER 4

SECTION 4B

CONCEPT CONNECTION

The Mild and Wild Amusement Park

Three friends, Travis, Kaitlyn, and Karsyn, spent the day at Mild and Wild Amusement Park, which features rides classified as Mild, Wild, or Super Wild. The park had two ticket packages as shown in the table.

Mild and Wild Amusement Park Ticket Packages		
Package	**Admission Fee**	**Ride Tickets**
Pick-ur-Tix	$5	Your choice at regular price
Mombo Combo	$5	8 of each type of ride at a 20% discount

The three friends chose the Pick-ur-Tix package. By the end of the day, Travis had ridden on 4 Mild rides, 8 Wild rides, and 8 Super Wild rides for a total ticket cost of $26. Kaitlyn had ridden on 8 Mild rides, 7 Wild rides, and 5 Super Wild rides for a total ticket cost of $24.25. Karsyn had ridden on 7 Mild rides, 6 Wild rides, and 4 Super Wild rides for a total ticket cost of $20.50.

1. Determine the ticket price for each type of ride. Solve an algebraic system for this situation by using matrices and a calculator or spreadsheet.
2. Determine the amount each person would spend if he or she had chosen the Mombo Combo. Explain which method of payment would have been bestfor each person.
3. Suppose that the amusement park had a fourth type of ride, called Colossal Wild. In addition to the other rides, Travis rode 12 Colossal Wild rides and spent $30. Kaitlyn rode 3 Colossal Wild rides and spent $30.25. Karsyn rode 1 Colossal Wild ride and spent $22.50. Would you be able to write and solve a matrix equation for this new situation? Explain.

READY TO GO ON?

Quiz for Lessons 4-4 Through 4-6

4-4 Determinants and Cramer's Rule

Find the determinant of each matrix.

1. $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} \frac{1}{2} & 0 \\ 3 & \frac{4}{5} \end{bmatrix}$ **3.** $\begin{bmatrix} 0.5 & 1.2 \\ -0.2 & 2.0 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & -1 & 3 \\ 0 & -2 & 1 \\ 4 & 4 & 1 \end{bmatrix}$

Use Cramer's rule to solve.

5. $\begin{cases} 2x + 3y = 5 \\ y = 1 - x \end{cases}$ **6.** $\begin{cases} x - y = 2 \\ y - x + 4 = 0 \end{cases}$ **7.** $\begin{cases} 2x - y + z = 3 \\ 3x + 2y = 2z + 1 \\ z = x + 2 \end{cases}$

4-5 Matrix Inverses and Solving Systems

Find the inverse of each matrix, if it is defined.

8. $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$ **9.** $\begin{bmatrix} -1 & \frac{1}{2} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$ **10.** $\begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$

Write the matrix equation for the system, and solve, if possible.

11. $\begin{cases} y = 2x - 1.5 \\ y - x = 0.5 \end{cases}$ **12.** $\begin{cases} 10x + 8y = 13 \\ 15x + 12y = 8 \end{cases}$ **13.** $\begin{cases} 5x + 7y = 3z + 3 \\ 3x + 4y = 6 - 2z \\ x + 3y = 5z - 7 \end{cases}$

14. You are writing three proposals for playground equipment as a system of equations. Use x as the price of a climbing wall, y as the price of a combination slide, and z as the price of an adventure maze. What is the price of each type of equipment?

$$\begin{cases} 2x + y + 3z = 23{,}650 \\ x + 3y + 2z = 20{,}450 \\ 3x + 2y + z = 24{,}600 \end{cases}$$

4-6 Row Operations and Augmented Matrices

Write the augmented matrix, and use row reduction to solve, if possible.

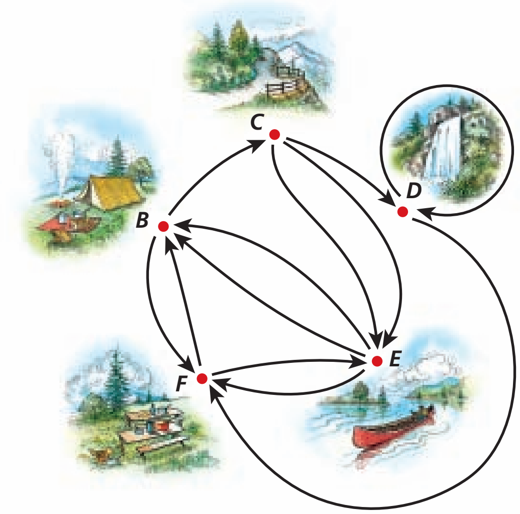
15. $\begin{cases} 2x + 5y = 5 \\ 50x = 30y + 1 \end{cases}$ **16.** $\begin{cases} 5x - 4y = 6 \\ 10x = 12 + 8y \end{cases}$ **17.** $\begin{cases} 6x + 5y + 8 = 0 \\ x - y = \frac{1}{2} \end{cases}$

18. The system of equations represents the costs of three fruit baskets. Use a to represent the cost of a pound of apples, b the cost of a pound of bananas, and g the cost of a pound of grapes. Find the cost of a pound of each type of fruit.

$$\begin{cases} 2a + 2b + g + 1.05 = 6.00 \\ 3a + 2b + 2g + 1.05 = 8.48 \\ 4a + 3b + 2g + 1.05 = 10.46 \end{cases}$$

EXTENSION

Networks and Matrices



Objective
Convert between finite graphs and their matrix representations, and calculate the number of trips via two vertices.

Vocabulary
adjacency matrix

A *network* is a finite set of connected points called *vertices*. A *directed network* is a network where arrows show the possible directions of travel between vertices, as in the figure shown. Networks represent connections in areas such as transportation, delivery routes, social interactions, and nature trails.

You can represent a network and show how many *1-step* (direct) paths are possible from each vertex to every other vertex by using an **adjacency matrix**.

EXAMPLE

Representing a Network with an Adjacency Matrix

In the network above, find the number of ways to go from C to F with exactly one stop in between (2-step paths).

First, write the adjacency matrix A that represents the network. This adjacency matrix shows the number of 1-step paths.

From: \ To:	B	C	D	E	F
B	0	1	0	0	1
C	0	0	1	2	0
D	0	0	1	0	1
E	2	0	0	0	1
F	1	0	0	1	0

Because there is a 1-step path (an arrow) from B to C, put a 1 in row 1 column 2.

Because there is no 1-step path (an arrow) from C to B, put a 0 in row 2 column 1.

The square of this adjacency matrix shows the number of 2-step paths (with one stop at a vertex in between).

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 4 & 0 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 2 \end{bmatrix}$$

A^2 shows that there are three 2-step paths from C to F. You can verify on the network that there are two paths from C to E to F and one path from C to D to F.

From: \ To:	B	C	D	E	F
B	1	0	1	3	0
C	4	0	1	0	3
D	1	0	1	1	1
E	1	2	0	1	2
F	2	1	0	0	2

As the network gets larger and more complex, this method helps you find the number of paths by calculating instead of by counting the paths on a graph.

CHECK IT OUT!

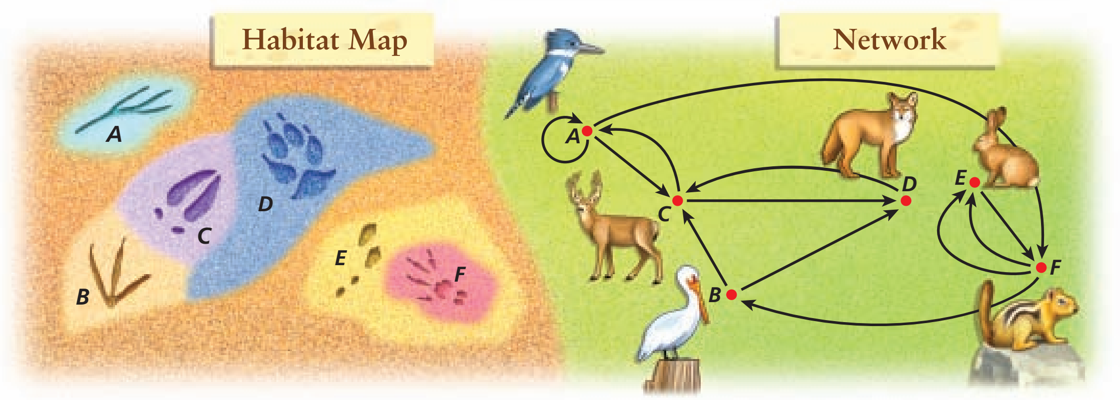
1. Use A^3 to show which vertex pairs in this network have five 3-step paths from one to the other.

EXTENSION

Exercises

Ecology **Use the following information for Exercises 1–5.**

Joel draws a habitat map and its network representation. The network vertices represent habitat patches, and the lines connecting them represent boundaries between the patches. The directed network shows wildlife migration patterns that Joel has recorded.



1. Write the adjacency matrix that represents the network shown. Keep the rows and columns in alphabetical order.
2. Find the number of ways to go from A to C.
   a. What is the number of 1-step paths?
   b. Show the matrix, and find the number of 2-step paths.
   c. Show the matrix, and find the number of 3-step paths.
   d. What is the total number of 1-, 2-, and 3-step paths from A to C?
3. Which two vertices are joined by exactly two 3-step paths?
4. Which two vertices are joined by exactly three 2-step paths?
5. For which vertices are 1-, 2-, or 3-step round-trips possible? How can you use the adjacency matrix to find the answer?

Helpful Hint

A *round-trip* path is a path that goes from one network vertex back to itself.

6. **Critical Thinking** What does an entry of 1 signify on the main diagonal of an adjacency matrix?
7. **Write About It** Explain how to represent a directed network with an adjacency matrix.
8. **Critical Thinking** What does an entry in the cube of an adjacency matrix tell you? What does a 0 entry in this matrix signify?
9. **ERROR ANALYSIS** A student said that the entry a_{mn} in an $n \times n$ adjacency matrix represents the number of paths from vertex n to vertex m. Explain the error.

Draw a directed network that can be represented by each adjacency matrix.

10. $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

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Complete the sentences below with vocabulary words from the list above.

1. A(n) ___?___ is a number that is multiplied by all entries of a matrix to form a new matrix.

2. A(n) ___?___ is formed from the constants in a system of equations.

3. Any matrix that has the same number of rows as columns is a(n) ___?___.

4-1 Matrices and Data *(pp. 246–252)*

Preparation for 2.0

EXAMPLE

$A = \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 9 \\ -7 & 8 \end{bmatrix}$

Evaluate, if possible.

■ $A - 2B$

$$= \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 9 \\ -7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -2(1) & -2(9) \\ -2(-7) & -2(8) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -2 & -18 \\ 14 & -16 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ 13 & -12 \end{bmatrix}$$

EXERCISES

$P = \begin{bmatrix} 3 & -5 & 2 \\ -4 & 1 & 3 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\quad R = \begin{bmatrix} 6 & -8 & 4 \\ -10 & 2 & 4 \end{bmatrix}$

Evaluate, if possible.

4. $P - 2Q$

5. $(0.2)Q$

6. $\frac{1}{2}R - \frac{1}{3}P$

7. $\frac{1}{2}(2P + R)$

Use the following data for Exercises 8–10.

At a beach cleanup, Ashton's team collected 125 cans and 45 bottles; Mark's team collected 95 cans and 65 bottles.

8. Display the data in the form of a matrix C.

9. Write matrix C_D to show team differences.

10. Each team received double its numbers in party points. Write matrix P to show the party points.

4-2 Multiplying Matrices *(pp. 253–260)*

Preparation for 2.0

EXAMPLES

Find the matrix product, if it is defined.

- $\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 7 & -5 \\ 0 & 1 & 0 \end{bmatrix}$

  $(2 \times 2)(2 \times 3)$

  $\begin{bmatrix} 2 & 7 & -5 \\ -6 & -19 & 15 \end{bmatrix}$

- $\begin{bmatrix} 5 & 1 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 4 & 16 \\ 0 & -2 \\ -12 & 1 \end{bmatrix}$

  $(2 \times 2)(3 \times 2)$

  undefined

- **Evaluate A^2, if possible.** $A = \begin{bmatrix} 3 & 4 & -5 \\ 0 & -2 & 7 \\ 9 & -6 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & 4 & -5 \\ 0 & -2 & 7 \\ 9 & -6 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & -5 \\ 0 & -2 & 7 \\ 9 & -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -36 & 34 & 8 \\ 63 & -38 & -7 \\ 36 & 42 & -86 \end{bmatrix}$$

EXERCISES

Find the matrix product, if it is defined.

$D = \begin{bmatrix} -1 & 2 \\ 0 & -2 \\ -3 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 0 & 1 & 3 \\ -2 & -1 & 4 \end{bmatrix}$ $F = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

11. DE **12.** FD **13.** DF **14.** EF

Evaluate, if possible.

15. D^2 **16.** F^2 **17.** $(ED)^2$

The tables show the prices and number of tickets sold for three theater performances.

	Adult	Student
Thu	$5	$2.50
Fri	$7.50	$4.25
Sat	$9	$5.75

	Thu	Fri	Sat
Adult	67	196	245
Student	104	75	154

18. a. Organize each table as a matrix.

b. Write the matrix product to find the amount of money collected for each performance.

c. Find the total collected for adult tickets and for student tickets for the three performances.

4-3 Using Matrices to Transform Geometric Figures *(pp. 261–267)*

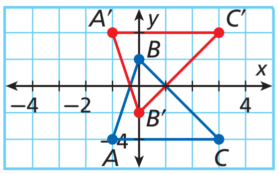
EXAMPLE

Use the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to transform triangle *ABC* with $A(-1, -2)$, $B(0, 1)$, and $C(3, -2)$. Graph the figure and its image. Describe the transformation.

- Multiply $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 \\ -2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 2 \end{bmatrix}$

The coordinates of the image are $A'(-1, 2)$, $B'(0, -1)$, and $C'(3, 2)$.

The triangle is reflected across the x-axis.



EXERCISES

Use matrices to transform polygon *P* with coordinates $W(-2, -1)$, $X(-1, 3)$, $Y(2, 4)$, and $Z(0, 0)$. Give the coordinates of each image.

19. Translate *P* 2 units right and 1 unit up.

20. Enlarge *P* by a factor of 1.5.

21. Use matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to transform *P*. Describe the transformation.

22. Use matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ to transform *P*. Describe the transformation.

4-4 Determinants and Cramer's Rule *(pp. 270–277)*

2.0

EXAMPLES

Find the determinant of each matrix.

■ $\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$

$\begin{vmatrix} 4 & -5 \\ 1 & 0 \end{vmatrix}$

$= 4(0) - 1(-5)$

$= 0 + 5 = 5$

■ $\begin{bmatrix} -\frac{1}{2} & 9 \\ \frac{2}{3} & -6 \end{bmatrix}$

$\begin{vmatrix} -\frac{1}{2} & 9 \\ \frac{2}{3} & -6 \end{vmatrix}$

$= -\frac{1}{2}(-6) - \frac{2}{3}(9)$

$= 3 - 6 = -3$

■ $\begin{bmatrix} 4 & 0 & 1 \\ 3 & 5 & -2 \\ 2 & -1 & 7 \end{bmatrix}$ write $\begin{vmatrix} 4 & 0 & 1 \\ 3 & 5 & -2 \\ 2 & -1 & 7 \end{vmatrix} \begin{matrix} 4 & 0 \\ 3 & 5 \\ 2 & -1 \end{matrix}$

$140 + (0) + (-3) - [10 + 8 + 0] = 137 - 18$

$D = 119$

Use Cramer's rule to solve each system of equations.

■ $\begin{cases} 3 + y = 3x \\ 5 - y = x \end{cases}$

Write in $ax + by = c$ form: $\begin{cases} 3x - y = 3 \\ x + y = 5 \end{cases}$

$D = \det\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = 3 - (-1) = 4$

$x = \dfrac{\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix}}{4} = \dfrac{8}{4} = 2 \quad y = \dfrac{\begin{vmatrix} 3 & 3 \\ 1 & 5 \end{vmatrix}}{4} = \dfrac{12}{4} = 3$

The solution is (2, 3).

■ $\begin{cases} 2a + 2b + c = 3 \\ -2a - 4b + 5c = 79 \\ a - 3b + 2c = 50 \end{cases} \rightarrow D = \begin{vmatrix} 2 & 2 & 1 \\ -2 & -4 & 5 \\ 1 & -3 & 2 \end{vmatrix} = 42$

$a = \dfrac{\begin{vmatrix} 3 & 2 & 1 \\ 79 & -4 & 5 \\ 50 & -3 & 2 \end{vmatrix}}{D} \quad b = \dfrac{\begin{vmatrix} 2 & 3 & 1 \\ -2 & 79 & 5 \\ 1 & 50 & 2 \end{vmatrix}}{D} \quad c = \dfrac{\begin{vmatrix} 2 & 2 & 3 \\ -2 & -4 & 79 \\ 1 & -3 & 50 \end{vmatrix}}{D}$

$a = \dfrac{168}{42} = 4 \quad b = \dfrac{-336}{42} = -8 \quad c = \dfrac{462}{42} = 11$

The solution is $a = 4, b = -8, c = 11$.

EXERCISES

Find the determinant of each matrix.

23. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

24. $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

25. $\begin{bmatrix} -\frac{1}{4} & 3 \\ -\frac{2}{3} & 6 \end{bmatrix}$

26. $\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

27. $\begin{bmatrix} 2 & 3 & -1 \\ -1 & 5 & 3 \\ 3 & -1 & -6 \end{bmatrix}$

28. $\begin{bmatrix} 3 & 2 & -1 \\ 5 & -3 & 2 \\ 9 & -13 & 8 \end{bmatrix}$

Use Cramer's rule to solve each system of equations.

29. $\begin{cases} x + y = 9 \\ x - y = 1 \end{cases}$

30. $\begin{cases} 2x + 5y + 21 = 0 \\ 6x = 47 + 7y \end{cases}$

31. $\begin{cases} 4.5x + 3y = 10.5 \\ 3x + 2y = 7 \end{cases}$

32. $\begin{cases} 5x - 6y = 7 + 7z \\ 6x - 4y + 10z = -34 \\ 2x + 4y = 29 + 3z \end{cases}$

33. $\begin{cases} x - y + z = 5 \\ y - x - z = 2 \\ x - y + z = 7 \end{cases}$

34. $\begin{cases} y - 2.4x = 0.8 \\ 3x + 0.5z = 2.25 \\ 3.5y + z = 8.5 \end{cases}$

35. Find the point of intersection of the lines given by the equations $2x + 3y = 8$ and $y = x + 1$.

a. Write the coefficient matrix, and find the determinant.

b. Solve using Cramer's rule.

36. At an end-of-season sale, a souvenir shop gave away small gifts valued at \$5 for sales of \$25 to \$74.99; medium gifts valued at \$8 for sales of \$75 to \$149.99; and large gifts valued at \$12.50 for sales above \$150. The store gave away 102 gifts worth a total of \$654 and six times as many small gifts as large gifts.

a. Write a system of equations for the situation.

b. Use Cramer's rule to solve for the number of small, medium, and large gifts.

4-5 Matrix Inverses and Solving Systems *(pp. 278–285)*

2.0

EXAMPLES

Find the inverse of the matrix, if it is defined.

- $A = \begin{bmatrix} 4 & -2 \\ 0 & -\frac{1}{2} \end{bmatrix}$

  $|A| = -2$; because $|A| \neq 0$, the matrix has an inverse.

  $\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ gives $\frac{1}{-2}\begin{bmatrix} -\frac{1}{2} & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -1 \\ 0 & -2 \end{bmatrix}$

  Check $\begin{bmatrix} 4 & -2 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{4} & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Write the matrix equation for the system. Solve.

- $\begin{cases} x + y = -6 \\ 2x + 3y = 8 \end{cases}$

  $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$ so $A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

  $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -6 \\ 8 \end{bmatrix}$ or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -26 \\ 20 \end{bmatrix}$

  The solution is $(-26, 20)$.

EXERCISES

Find the inverse of the matrix, if it exists.

37. $\begin{bmatrix} 6 & 2 \\ -1 & 3 \end{bmatrix}$

38. $\begin{bmatrix} \frac{3}{4} & -\frac{2}{5} \\ 0 & \frac{1}{5} \end{bmatrix}$

39. $\begin{bmatrix} 2 & 5 \\ 1 & 2.5 \end{bmatrix}$

40. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

41. $\begin{bmatrix} -1.5 & 1 & 0.5 \\ 0.5 & 1 & 1 \\ -1 & 1 & 0.5 \end{bmatrix}$

42. $\begin{bmatrix} 5 & -3 & 2 \\ 0 & 0 & 0 \\ 2 & 7 & -1 \end{bmatrix}$

Write the matrix equation for the system. Solve.

43. $\begin{cases} \frac{3}{2}x = 20 + y \\ x + 6y = 80 \end{cases}$

44. $\begin{cases} x = 1 + y \\ x + y = 9 \end{cases}$

45. $\begin{cases} 3x + 3y = 19 + z \\ 5x + 4y - 28 = 2z \\ 2(x + y) - 12 = z \end{cases}$

46. $\begin{cases} 2x + 9 = 2z \\ 5x + y + 32 = 7z \\ 2(3x + y) = 8z - 39 \end{cases}$

4-6 Row Operations and Augmented Matrices *(pp. 287–293)*

2.0

EXAMPLE

Write the augmented matrix, and solve.

- $\begin{cases} x - y = 3 \\ x - y = 0 \end{cases}$

  $\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 1 & -1 & 0 \end{array}\right]$ ❶ − ❷ → $\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 3 \end{array}\right]$

The second row means $0 + 0y = 3$, which is false. The system is inconsistent.

- $\begin{cases} 2x + y = 6 \\ x - y = 0 \end{cases}$

  $\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & -1 & 0 \end{array}\right]$ (❶ + ❷) ÷ 3 → $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & -1 & 0 \end{array}\right]$

  ❶ − ❷ → $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right]$, so $x = 2$ and $y = 2$.

EXERCISES

Write the augmented matrix, and solve.

47. $\begin{cases} 7x + 2y = 0.75 \\ 2x - y = 1 \end{cases}$

48. $\begin{cases} p - q = 4 \\ 2p + 3q = -22 \end{cases}$

Solve the system by using row reduction.

49. $\begin{cases} x + 2z = 0.5 \\ -5y = 0.25 \\ 3x + 4z = 1.1 \end{cases}$

50. $\begin{cases} 2.5x + 1.5y = 4 \\ 3.2x + y = 4z - 3.8 \\ 6.4x - 5y + 2.1z = 5.6 \end{cases}$

51. In gymnastics, Team Osho won 27 awards, which gave them 87 points. The team won one more 1st-place award than 3rd-place awards.

Place	Points
First	5
Second	4
Third	1

Use the table to write a system of equations to represent this situation. Use row reduction to find how many of each award the team won.

CHAPTER 4

CHAPTER TEST

Use the data from the table to answer the questions.

1. Display the data in the form of matrix A.
2. What are the dimensions of the matrix?
3. What is the value of the matrix entry with address a_{31}?
4. What is the address of the entry that has a value of 2?

Awards Given

	First Place	Second Place	Third Place	Total Points
Klete	5	1	2	41
Michael	3	5	1	42
Ryan	3	1	4	29

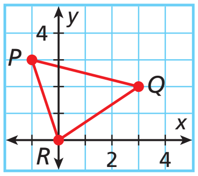
Evaluate, if possible.

$$E=\begin{bmatrix}2 & 3\\ -1 & 0\\ 4 & 1\end{bmatrix} \quad F=\begin{bmatrix}4 & -2 & 0\\ -1 & 1 & -2\end{bmatrix} \quad G=\begin{bmatrix}2 & -1\\ 3 & 1\end{bmatrix} \quad H=\begin{bmatrix}-2 & 1\\ 3 & 0\\ 5 & -1\end{bmatrix} \quad J=\begin{bmatrix}1 & -5 & 6\end{bmatrix} \quad K=\begin{bmatrix}7\\ 0\\ -2\end{bmatrix}$$

5. $E + F$
6. EF
7. FE
8. H^2
9. G^3
10. FK

Use a matrix to transform $\triangle PQR$.

11. Translate $\triangle PQR$ 2 units up and 1 unit right.
12. Enlarge $\triangle PQR$ by a factor of $\frac{3}{2}$.
13. Use $\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}$ to transform $\triangle PQR$. Describe the image.



Find the determinant of each matrix.

14. $\begin{bmatrix}4 & 0\\ 0 & -3\end{bmatrix}$
15. $\begin{bmatrix}0.25 & 1\\ 2 & 8\end{bmatrix}$
16. $\begin{bmatrix}3 & -1\\ -2 & -1\end{bmatrix}$
17. $\begin{bmatrix}1 & -2 & 3\\ 3 & -1 & -3\\ 2 & 1 & 5\end{bmatrix}$
18. Use Cramer's rule to solve $\begin{cases}x + 2y = 1\\ 3x - y = 10\end{cases}$
19. Use Cramer's rule to solve $\begin{cases}x + 3z = 3 + 2y\\ 3x + 22 = y + 3z\\ 2x + y + 5z = 8\end{cases}$

Find the inverse, if it exists.

20. $\begin{bmatrix}2 & 0.7\\ 4 & 1.4\end{bmatrix}$
21. $\begin{bmatrix}3 & -1\\ 1 & 3\end{bmatrix}$
22. $\begin{bmatrix}3 & 1\\ 2 & -1\end{bmatrix}$
23. $\begin{bmatrix}3 & 2 & -1\\ 2 & 3 & -5\\ 1 & 4 & 2\end{bmatrix}$
24. The cost of 2.5 pounds of figs and 1.5 pounds of dates is \$14.42. The cost of 3.5 pounds of figs and 1 pound of dates is \$16.91. Use a matrix operation to find the price of each per pound.

Write the matrix equation for each system, and solve, if possible.

25. $\begin{cases}6x + y = 2\\ 3x - 2y + 1 = 0\end{cases}$
26. $\begin{cases}5x - 2y = 3\\ 2.5x - y = 1.5\end{cases}$
27. $\begin{cases}x + 2y = 3.5\\ 3x = 2.7 + y\end{cases}$
28. $\begin{cases}2x - z = 3 + y\\ x + 2 = y + 5\\ 4z + x + y = 1\end{cases}$

Write the augmented matrix, and use row reduction to solve, if possible.

29. Use the data from Items 1–4 above. Find the number of points assigned for finishing in first, second, and third places.

College Entrance Exam Practice

FOCUS ON SAT MATHEMATICS SUBJECT TEST

There are two levels of SAT Mathematics Subject Tests: Level 1 and Level 2. Each test has 50 questions, all multiple choice. The content of each test is very different. Getting a high score on one test does not mean you will get a high score on the other test.

You may want to time yourself as you take this practice test. It should take you about 6 minutes to complete.

HOT TIP! You can write all over the test book to sketch figures, do scratch work, or cross out incorrect answers to help you eliminate choices. Remember to mark your final answer on the answer sheet because the test books are not examined for answers.

1. If A is a 6×4 matrix and B is a 4×8 matrix, what are the dimensions of matrix AB?

(A) 6×4

(B) 4×8

(C) 10×12

(D) 12×8

(E) 6×8

2. If $D = \begin{bmatrix} 5 & 1 \\ 8 & 3 \\ 6 & 2 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & -5 \\ 1 & 4 \\ -2 & 3 \end{bmatrix}$, which of the following operations gives $\begin{bmatrix} 5 & 11 \\ 6 & -5 \\ 10 & -4 \end{bmatrix}$?

(A) $D + 2E$

(B) $D - 2E$

(C) $2D + E$

(D) $2D - E$

(E) $D + E$

3. Given a matrix representing a system of equations, which of the following row operations is NOT valid for solving the system?

(A) Add the first row to the second row.

(B) Multiply the last row by -1.

(C) Switch the top row and the bottom row.

(D) Subtract the second row from the first row.

(E) Add 1 to each element of the last row.

4. Which of the following matrices has a determinant of 3?

(A) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$

(E) $\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$

5. What effect does adding the matrix $\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ to a matrix representing ordered pairs on a line have?

(A) The line is translated 3 units to the right and 1 unit down.

(B) The line is translated 3 units to the left and 1 unit up.

(C) The line is translated 1 unit to the left and 3 units up.

(D) The line is stretched and rotated 90° clockwise.

(E) The line is stretched and rotated 90° counterclockwise.

CHAPTER 4

Strategies for Success

Extended Response: Write Extended Responses

Extended response test items evaluate how well you can apply and explain mathematical concepts. These questions have multiple parts, and you must correctly answer all of the parts to receive full credit. Extended response questions are scored using a 4-point scoring system.

Scoring Rubric

4 points: The student demonstrates a thorough understanding of the concept, correctly answers the question, and provides a complete explanation.

3 points: The student shows most of the work and provides an explanation but has a minor computation error, OR student shows all work and arrives at a correct answer but does not provide an explanation.

2 points: The student makes major errors resulting in an incorrect solution.

1 point: The student shows no work and has an incorrect response, OR student does not follow directions.

0 points: The student gives no response.

EXAMPLE 1

Extended Response An amphitheater has two levels of seating for concerts. Upper-level tickets sell for \$25, and lower-level tickets sell for \$50. At the last concert, 220 tickets were sold, and \$7875 was taken in. How many tickets of each type were sold? Use a system of linear equations to model this situation. Use matrices to solve the system of equations. Interpret your results.

The following shows a response that received **4 points.** Notice that it includes a system of equations that models the situation with variables clearly defined, matrix operations, the final matrix, and a correct solution written in a complete sentence.

Let u = the number of upper-level tickets sold.
Let ℓ = the number of lower-level tickets sold. ← Variables defined

System of Equations

$$\begin{cases} u + \ell = 220 \\ 25u + 50\ell = 7875 \end{cases}$$

Matrix Equation

$$\begin{bmatrix} 1 & 1 \\ 25 & 50 \end{bmatrix} X = \begin{bmatrix} 220 \\ 7875 \end{bmatrix}$$

Inverse

$$\frac{1}{25}\begin{bmatrix} 50 & -1 \\ -25 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -0.04 \\ -1 & 0.04 \end{bmatrix}$$

$X = A^{-1}B$

$$\begin{bmatrix} 2 & -0.04 \\ -1 & 0.04 \end{bmatrix}\begin{bmatrix} 220 \\ 7875 \end{bmatrix} = \begin{bmatrix} 2(220) - 0.04(7875) \\ -1(220) + 0.04(7875) \end{bmatrix} = \begin{bmatrix} 125 \\ 95 \end{bmatrix}$$

Solution: There were 125 upper-level tickets and 95 lower-level tickets sold.

HOT TIP! Highlight or underline each part of the test item. Verify that your response addresses each part of the problem before you move on.

Read each test item, and answer the questions that follow.

Item A

Extended Response Explain what row operations were performed to create the new matrix.

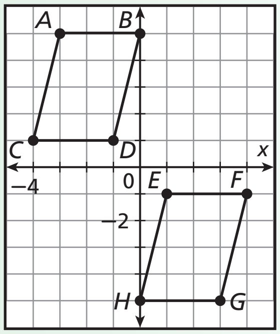
$$\left[\begin{array}{cc|c} 4 & 3 & 1 \\ 3 & -2 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -17 & 0 & -17 \\ 7 & 1 & 6 \end{array}\right]$$

1. Tyler wrote this response:

> First, add row 1 to row 2 to get a new row 2. This makes a new matrix, $\begin{bmatrix} 4 & 3 & 1 \\ 7 & 1 & 6 \end{bmatrix}$. Then multiply the new row 2 by -3 and add it to row 1. This makes the resulting matrix, $\begin{bmatrix} -17 & 0 & -17 \\ 7 & 1 & 6 \end{bmatrix}$.

Item B

Extended Response Write the matrix that represents the vertices of figure *ABCD*. Then, determine and explain what matrix and which operations you would use to create figure *EFGH*.



2. Make a list of what needs to be included in a response to this test item so that it receives full credit.

3. Sarah wrote this response:

> The matrix for figure ABCD is $\begin{bmatrix} -3 & 0 & -1 & -4 \\ 5 & 5 & 1 & 1 \end{bmatrix}$, where the columns represent points A, B, C, and D and the rows represent the x- and y-coordinates.

Score Sarah's response, and provide your reasoning for the score.

4. Give a response that would receive full credit.

Item C

Extended Response Create a matrix that does not have an inverse. Explain your reasoning.

5. Should the following response receive full credit? Explain your reasoning.

> The matrix $\begin{bmatrix} 4 & 3 \\ 12 & 9 \end{bmatrix}$ does not have an inverse because its determinant equals zero.
>
> $\begin{vmatrix} 4 & 3 \\ 12 & 9 \end{vmatrix} = (4 \cdot 9) - (12 \cdot 3) = 36 - 36 = 0$

Item D

Extended Response Consider the system of equations $\begin{cases} x - 4y = 1.5 \\ 2x + y = 8.2 \end{cases}$. Describe which method—row operations, inverse matrices, or Cramer's rule—you would use to solve the system. Explain your reasoning.

6. Score the following response, and explain your score.

> $D = \begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} = 1 - (-8) = 9$
>
> $x = \dfrac{\begin{vmatrix} 1.5 & -4 \\ 8.2 & 1 \end{vmatrix}}{9} = \dfrac{1.5 - (-32.8)}{9} = \dfrac{34.3}{9} \approx 3.81$
>
> $y = \dfrac{\begin{vmatrix} 1 & 1.5 \\ 2 & 8.2 \end{vmatrix}}{9} = \dfrac{8.2 - 3}{9} = \dfrac{5.2}{9} \approx 0.58$

7. How would you rewrite this response so that it receives full credit?

CHAPTER 4

STANDARDIZED TEST PREP

go.hrw.com
Test Practice Online
KEYWORD: MB7 TestPrep

CUMULATIVE ASSESSMENT, CHAPTERS 1–4

Multiple Choice

1. Jack is two less than four times Macy's age. Kirstin is six more than half of Jack's age. If *x* is Macy's age and *y* is Jack's age, which expression represents Kirstin's age?

Ⓐ $\frac{1}{2}x + 6$

Ⓑ $2x + 5$

Ⓒ $4x + \frac{1}{2}y + 4$

Ⓓ $\frac{1}{2}(4x + 2) - 6$

2. The matrix below is the augmented matrix for a system of equations. What is the solution of the system of equations?

$$\left[\begin{array}{cc:c} 6 & 8 & 5 \\ 12 & 4 & 16 \end{array}\right]$$

Ⓕ $\left(-\frac{3}{2}, \frac{1}{2}\right)$

Ⓖ $\left(-\frac{1}{2}, \frac{3}{2}\right)$

Ⓗ $\left(\frac{2}{3}, -2\right)$

Ⓙ $\left(\frac{3}{2}, -\frac{1}{2}\right)$

3.

PUTTER'S MINIATURE GOLF

1 putt—HOLE-IN-ONE!
2 putts—BIRDIE
3 putts—PAR
4 putts—BOGEY

Grace played 18 holes of miniature golf. On each hole, she made a birdie, a par, or a bogey. She made four more pars than birdies and bogeys combined. Her total score was 55. How many birdies did Grace get?

Ⓐ 3

Ⓑ 4

Ⓒ 7

Ⓓ 11

4. When stopping a car, a driver takes about 1.5 seconds to react before beginning to brake. A car traveling at 30 miles per hour moves 66 feet before the driver's foot touches the brake pedal. A car traveling at 45 miles per hour moves 99 feet, and a car traveling at 55 miles per hour moves 121 feet. Which set consists of only domain values for the given data?

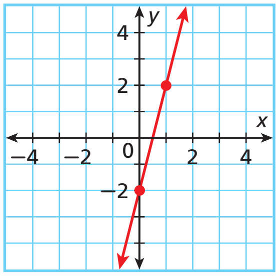
Ⓕ $\{1.5\}$

Ⓖ $\{30, 66\}$

Ⓗ $\{30, 45\}$

Ⓙ $\{66, 99, 121\}$

5. The graph below shows the graph of an equation that is the boundary line of an inequality. The ordered pairs (21, 83) and (16, 62) are NOT solutions of the inequality. Which of these is true of the graph of the inequality?



Ⓐ The boundary line should be dashed, and the half-plane above the line should be shaded.

Ⓑ The boundary line should be solid, and the half-plane above the line should be shaded.

Ⓒ The boundary line should be dashed, and the half-plane below the line should be shaded.

Ⓓ The boundary line should be solid, and the half-plane below the line should be shaded.

6. Which matrix expression results in the matrix $\begin{bmatrix} 2 & -4 \\ 11 & 14 \end{bmatrix}$?

Ⓕ $\frac{1}{2}\begin{bmatrix} 4 & -8 \\ 22 & 28 \end{bmatrix}$

Ⓖ $2\begin{bmatrix} 0 & -6 \\ 9 & 12 \end{bmatrix}$

Ⓗ $\begin{bmatrix} 2 & -4 \\ 11 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ⓙ $\begin{bmatrix} -6 & 17 \\ 8 & 10 \end{bmatrix} + \begin{bmatrix} 8 & -13 \\ -3 & 4 \end{bmatrix}$

7. On March 27, 2004, NASA's hypersonic research aircraft X-43A reached a speed of Mach 7. Traveling at seven times the speed of sound, an aircraft moves 16 miles every 12 seconds. Which of these functions represents the number of miles an aircraft traveling at Mach 7 can go in *s* seconds?

(A) $f(s) = 16x + 12s$

(B) $f(s) = \frac{3}{4}s$

(C) $f(s) = 16s$

(D) $f(s) = 1\frac{1}{3}s$

HOT TIP! In order for the correlation coefficient to be displayed when you calculate the linear regression, your calculator should be set to **DiagnosticOn**.

8. After a conference, Brent was asked to rate each of the five workshops he attended on a scale from 1 to 10. The table below shows the length and Brent's rating of each workshop.

Minutes	53	93	48	120	32
Rating	7	4	5	9	8

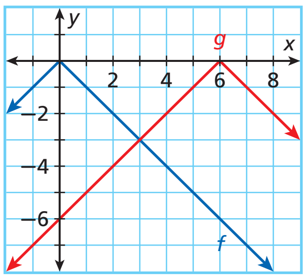
What is the correlation coefficient, rounded to the nearest hundredth, for the relationship between the length and Brent's rating of each workshop?

(F) 0.01 (H) 0.88

(G) 0.12 (J) 6.13

Gridded Response

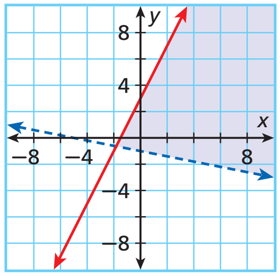
9. Examine the graphs of $f(x) = -|x|$ and $g(x) = f(x - h)$. What is the value of h?



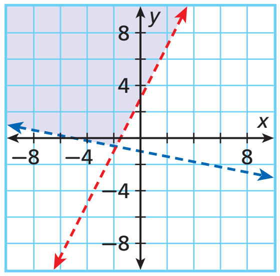
10. Find the determinant of the matrix $\begin{bmatrix} \frac{2}{5} & -1 \\ 0.4 & 10 \end{bmatrix}$.

Short Response

11. a. Name the system of inequalities.



b. Name the system of inequalities.



c. Describe how the system in part **a** differs from the system in part **b.**

12. Use a matrix and $\triangle ABC$ with coordinates $(-1, 0)$, $(4, 3)$, and $(2, -1)$ for the transformations.

a. Translate $\triangle ABC$ 1 unit to the right and 4 units up. Give the coordinates of $\triangle A'B'C'$.

b. Reflect $\triangle A'B'C'$ across the *y*-axis. Give the coordinates of $\triangle A''B''C''$.

Extended Response

13. Use the linear function $2x - 3y = -15$.

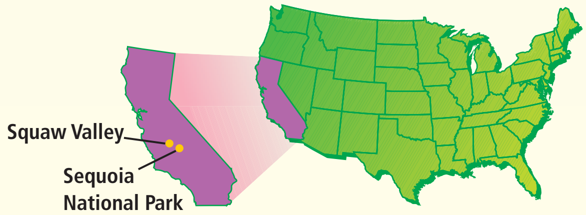
a. Explain how to rewrite the equation in slope-intercept form.

b. Describe why the slope-intercept form is usually the best way to write an equation before you graph it.

c. Write a step-by-step explanation on how to graph the equation.

Problem Solving on Location

CALIFORNIA



Squaw Valley

Squaw Valley, located in the Sierra Nevadas and more than a mile above sea level, has been a scenic paradise for recreation in both summer and winter since 1949. Hiking, climbing, mountain biking, skating, and sleigh rides are often available, but Squaw Valley is best known for its downhill skiing.

Choose one or more strategies to solve each problem. For 1 and 2, use the table.

One Day Lift Tickets Purchased

	Adult	Youth (Age 13–18)	Seniors	Total Cost ($)
Group A	9	7	3	1111
Group B	6	11	1	1028
Group C	2	12	0	762

1. The table shows how many lift tickets three different groups of skiers purchased. Determine the cost for a group of 1 adult and 7 seniors.

2. Evening rates are applied between 4 P.M. and 9 P.M. Had evening rates been charged to groups A, B, and C, they would have paid $321, $297, and $220, respectively. An evening group of 2 adults, 3 youths, and 1 senior was incorrectly charged the full rate. How much should the group be refunded?

3. The cable car ride from Squaw Valley's base area (elevation 6200 feet) to High Camp (elevation 8200 feet) takes 8 minutes. At what speed, in mi/h, does the cable car rise?

4. Ice skating at the Olympic Ice Pavillion, where in 1960 the United States won its first gold medal in hockey, is $10 without cable car rides and $25 with cable car rides. During the first ten minutes of a skating session, 28 people paid a total of $460 to skate. How many of those people also bought cable car rides?

Problem Solving Strategies

- Draw a Diagram
- Make a Model
- Guess and Test
- Work Backward
- Find a Pattern
- Make a Table
- Solve a Simpler Problem
- Use Logical Reasoning
- Use a Venn Diagram
- Make an Organized List

General Sherman Tree

The General Sherman Tree, located in Sequoia National Park, is the largest tree in the world by volume. In 1975, the General Sherman Tree was determined to have a volume of greater than 52,500 ft^3.

Choose one or more strategies to solve each problem.

Tree enthusiasts use a point system to compare trees. They assign a specific number of points for each inch of circumference, each foot of height, and each foot of crown spread. The table shows the measurements and point totals, rounded to the nearest tenth, for several species of California's champion trees.

For 1, use the table.

1. How many points should be assigned to Jasper, California's national champion coast live oak, which has a circumference of 338 inches, a height of 58 ft, and a crown spread of 75 ft?

California Champion Trees

	Circumference (in.)	Height (ft)	Crown Spread (ft)	Total Points
General Sherman	998	275	106.5	1299.6
California Sycamore	344	104	94	475.1
Methuselah	473	47	41	530.3

2. The combined volume of a stack of cylinders with different radii and heights can be found by using the matrix equation shown.

$$\pi\begin{bmatrix} r_1^2 & r_2^2 & \cdots \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ \vdots \end{bmatrix}$$

Use cylinders to model the General Sherman Tree. Use the midpoints of the heights given in the table below to find the height of each cylinder. Estimate the volume of the tree to the nearest cubic foot.

General Sherman Measurements

Height	Diameter (ft)
maximum at base	36.5
60 ft above ground	17.5
180 ft above ground	14

3. Because the General Sherman Tree narrows sharply above the base, the volume is more accurately estimated without using the base diameter. Revise your estimate of the volume of the General Sherman Tree, and compare the new volume to your answer in Problem 2.